Solutions to Folland

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(a) Let $\mathcal{M} \subset \mathcal{P}(X)$ be a σ-algebra. If $E \in \mathcal{M}$ is non-empty it is easy to verify that $\mathcal{M}(E) = \{A \cap E : A \in \mathcal{M}\}$ is a σ-algebra. Furthermore, since $\mathcal{M}$ is infinite, $\mathcal{M} \neq \{\emptyset, X\}$ so there is some $E_1 \in \mathcal{M}$ such that neither E_1 nor E_1^c is empty. Furthermore, at least one of $\mathcal{M}(E_1)$ or $\mathcal{M}(E_1^c)$ must be infinite since the map

$$\varphi : \mathcal{M} \to \mathcal{M}(E_1) \times \mathcal{M}(E_1^c) \text{ which sends } \varphi : A \mapsto (A \cap E_1, A \cap E_1^c)$$

is injective (in particular $|\mathcal{M}| \leq |\mathcal{M}(E_1) \times \mathcal{M}(E_1^c)| = |\mathcal{M}(E_1)| \cdot |\mathcal{M}(E_1^c)|$). Suppose without loss of generality that it is $\mathcal{M}(E_1^c)$ which is infinite. Then inductively choose E_i from $\mathcal{M}(E_{i-1}^c)$ so that $\mathcal{M}(E_i^c)$ is infinite. We have that $E_i \cap E_{i-1} = \emptyset$ since $E_i \subset E_{i-1}^c$, and $\mathcal{M}(E_i^c) \subset \mathcal{M}$. Thus $E_1, E_2, \ldots$ constructed in this way is a sequence of disjoint, non-empty sets in $\mathcal{M}$.

(b) The map $\varphi : 2^{\mathbb{N}} \to \mathcal{M}$ which sends $\varphi : \mathcal{I} \mapsto \bigcup_{i \in \mathcal{I}} E_i$ is injective. Indeed

$$\varphi(\mathcal{I}) = \varphi(\mathcal{J}) \implies \bigcup_{i \in \mathcal{I}} E_i = \bigcup_{i \in \mathcal{J}} E_i \implies \mathcal{I} = \mathcal{J}$$

since the E_i are disjoint. Thus $|\mathcal{M}| \geq |2^{\mathbb{N}}| = |\mathbb{R}|$.

Folland 1.4. If $\mathcal{A}$ is a σ-algebra there is nothing to prove. Suppose $\mathcal{A}$ is an algebra that is closed under countable increasing unions. Let $\{A_n\}_{n \in \mathbb{N}}$ be in $\mathcal{A}$ and define $B_n = \bigcup_{i=1}^n A_i$ so that $B_1 \subset B_2 \subset \cdots$. Then $B_n \in \mathcal{A}$ since $\mathcal{A}$ is an algebra (closed under finite union). Furthermore using closure under countable increasing unions:

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in \mathcal{A}.$$

Folland 1.5. Let $\mathcal{E}$ be a collection of sets and denote $\sigma\mathcal{E}$ the σ-algebra generated by $\mathcal{E}$, we show that $\mathcal{M} = \sigma\mathcal{E}$ where $\mathcal{M}$ is defined as

$$\mathcal{M} = \bigcup_{\substack{\mathcal{F} \subset \mathcal{E} \\ \mathcal{F} \text{ countable}}} \sigma\mathcal{F}.$$

First $\mathcal{M}$ is a σ-algebra. Indeed if $A \in \mathcal{M}$, then $A \in \sigma\mathcal{F}$ for some countable $\mathcal{F} \subset \mathcal{E}$. Thus $A^c \in \sigma\mathcal{F}$ and so $A^c \in \mathcal{M}$. Now let $\{A_n\}_{n \in \mathbb{N}}$ be in $\mathcal{M}$. Then each $A_n \in \sigma\mathcal{F}_n$ for some countable $\mathcal{F}_n \subset \mathcal{E}$. Define $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$. Then for each $n \in \mathbb{N}$ we have $\mathcal{F}_n \subset \mathcal{F}$, and thus $\sigma\mathcal{F}_n \subset \sigma\mathcal{F}$. In particular $A_n \in \sigma\mathcal{F}$ for each $n \in \mathbb{N}$ and so $\bigcup_{n \in \mathbb{N}} A_n \in \sigma\mathcal{F}$. Furthermore, $\mathcal{F}$ is a countable union of countable sets and is hence countable, in particular $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$.

Now $\mathcal{M}$ contains $\mathcal{E}$. Indeed if $E \in \mathcal{E}$, then $E \in \sigma\{E\} \subset \mathcal{M}$. By minimality it follows that $\sigma\mathcal{E} \subset \mathcal{M}$. On the other hand, if $E \in \mathcal{M}$ then there exists $\mathcal{F} \subset \mathcal{E}$ such that $E \in \sigma\mathcal{F}$. But $\sigma\mathcal{F} \subset \sigma\mathcal{E}$, so $E \in \sigma\mathcal{E}$. Thus $\mathcal{M} \subset \sigma\mathcal{E}$ and altogether we have $\mathcal{M} = \sigma\mathcal{E}$.

Folland 1.6. As in Theorem 1.9, let $(X, \mathcal{M}, \mu)$ be a measure space, $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and

$$\overline{\mathcal{M}} = \{E \cup Z : E \in \mathcal{M} \text{ and } Z \subset N \text{ for some } N \in \mathcal{N}\}.$$

It has been shown that $\overline{\mathcal{M}}$ is a σ-algebra. We show that the extension $\overline{\mu}(E \cup Z) = \mu(E)$ is a complete measure on $\overline{\mathcal{M}}$. First that $\overline{\mu}$ is a measure. First of all $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$. Now let $\{A_n\}_{n \in \mathbb{N}}$ be disjoint in $\overline{\mathcal{M}}$. Then $A_n = E_n \cup Z_n$ for some $E_n \in \mathcal{M}$ and $Z_n \subset N_n$ where $N_n \in \mathcal{N}$. Then (noting $\bigcup_{n \in \mathbb{N}} N_n$ is a null set)

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} E_n \cup \underbrace{\bigcup_{n \in \mathbb{N}} Z_n}_{\subset \bigcup_{n \in \mathbb{N}} N_n},$$

and since the E_n must be disjoint (otherwise the A_n would not be disjoint)

$$\overline{\mu}\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} \mu(E_n) = \sum_{n \in \mathbb{N}} \overline{\mu}(A_n).$$

Now we show $\overline{\mu}$ is complete. Let $A = E \cup Z$ (where $Z \subset N$ and $N \in \mathcal{N}$) be a null set in $\overline{\mathcal{M}}$, then $0 = \overline{\mu}(A) = \mu(E)$. In particular E is a null set in $\mathcal{M}$. Furthermore, $E \cup N$ is a null set containing A. Now take an $S \subset A$, since $\emptyset \in \mathcal{M}$ and $S \subset E \cup N \in \mathcal{N}$ we can write $S = \emptyset \cup S$ and so $\overline{\mu}(S) = \mu(\emptyset) = 0$.

Folland 1.7. Define $\mu = \sum_{j=1}^n a_j \mu_j$. Then $\mu(\emptyset) = \sum_{j=1}^n a_j \mu_j(\emptyset) = \sum_{j=1}^n a_j 0 = \sum_{j=1}^n 0 = 0$. Furthermore, for disjoint $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}$ we can use Fubini to write

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{j=1}^n a_j \mu_j \left(\sum_{i=1}^{\infty} A_i\right) = \sum_{j=1}^n a_j \sum_{i=1}^{\infty} \mu_j(A_i) = \sum_{i=1}^{\infty}\sum_{j=1}^{n} \mu_j(A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

Folland 1.8. Recall the definitions of $\limsup$ and $\liminf$ for a sequence of sets $\{E_n\}_{n\in\mathbb{N}}$:

$$\limsup_n E_n = \bigcap_{n\geq 1}\bigcup_{k\geq n} E_k \qquad \text{and} \qquad \liminf_n E_n = \bigcup_{n\geq 1}\bigcap_{k\geq n} E_k.$$

Define the sets

$$A_n = \bigcup_{k\geq n} E_k \qquad \text{and} \qquad B_n = \bigcap_{k\geq n} E_k.$$

First of all $B_n \subset E_k$ for all $k \geq n$, thus $\mu(B_n) \leq \mu(E_k)$ for all $k \geq n$, and taking the infimum yields

$$\mu(B_n) \leq \inf_{k\geq n} \mu(E_k).$$

Furthermore notice that $B_1 \subset B_2 \subset \cdots$, and so by continuity from below:

$$\mu\left(\liminf_n E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n) \leq \lim_{n\to\infty}\inf_{k\geq n} \mu(E_k) = \liminf_{n\to\infty} \mu(E_n).$$

On the other hand $A_n \supset E_k$ for all $k \geq n$, thus $\mu(A_n) \geq \mu(E_k)$ for all $k \geq n$, and taking the supremum yields

$$\mu(A_n) \geq \sup_{k\geq n} \mu(E_k).$$

Now since $A_1 \supset A_2 \supset \cdots$ and by assumption $\mu(A_1) < \infty$ we use continuity from above to write:

$$\mu\left(\limsup_n E_n\right) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n) \geq \lim_{n\to\infty}\sup_{k\geq n} \mu(E_k) = \limsup_{n\to\infty} \mu(E_n).$$

Folland 1.9. We can write the disjoint unions

$$E = (E\cap F)\cup(E\cap F^c) \qquad F = (F\cap E)\cup(F\cap E^c) \qquad E\cup F = (E\cap F^c)\cup(F\cap E^c)\cup(E\cap F).$$

Hence

$$\mu(E) = \mu(E\cap F)+\mu(E\cap F^c) \qquad \mu(F) = \mu(F\cap E)+\mu(F\cap E^c) \qquad \mu(E\cup F) = \mu(E\cap F^c)+\mu(F\cap E^c)+\mu(E\cap F).$$

Finally we can write:

$$\mu(E)+\mu(F) = \mu(E\cap F^c)+\mu(F\cap E^c)+\mu(E\cap F)+\mu(E\cap F) = \mu(E\cup F)+\mu(E\cap F).$$

Folland 1.10. First of all $\mu_E(\emptyset) = \mu(\emptyset\cap E) = \mu(\emptyset) = 0$. Now let $\{A_n\}_{n\in\mathbb{N}}$ be disjoint sets in $\mathcal{M}$. Then

$$\mu_E\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(E\cap\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty}(A_n\cap E)\right) = \sum_{n=1}^{\infty}\mu(A_n\cap E) = \sum_{n=1}^{\infty}\mu_E(A_n).$$

Note that $(A_n\cap E)\cap(A_m\cap E) = (A_n\cap A_m)\cap E = \emptyset\cap E = \emptyset$ for $m \geq n$.

Folland 1.11. If μ is a measure then there is nothing to prove. So suppose μ is a finitely additive measure and is continuous from below. Let $\{A_n\}_{n\in\mathbb{N}}$ be disjoint sets in $\mathcal{M}$. Define $B_n = \bigcup_{i=1}^n$ so that $B_1 \subset B_2 \subset \cdots$. Then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty}\mu(B_n) = \lim_{n\to\infty}\sum_{i=1}^{n}\mu(A_i) = \sum_{i=1}^{\infty}\mu(A_i).$$

On the other hand suppose μ is finitely additive and continuous from above, it suffices to show that μ is continuous from below. Let $E_1 \subset E_2 \subset \cdots$, and define $F_n = X \setminus E_n$ so that $\mu(F_n) = \mu(X) - \mu(E_n)$. Then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} (X \setminus F_n)\right) = \mu(X) - \mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \mu(X) - \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu(E_n).$$

Folland 1.12. Recall the notation $A\triangle B := (A \setminus B) \cup (B \setminus A)$ is the symmetric difference

(a) We have
$$0 = \mu(E\triangle F) = \mu((E \setminus F) \cup (F \setminus E)) = \mu(E \setminus F) + \mu(F \setminus E).$$
Hence both
$$\mu(E \setminus F) = \mu(F \setminus E) = 0.$$
In particular
$$\mu(E) = \mu(E \cap F) + \mu(E \setminus F) = \mu(E \cap F) = \mu(F \cap E) + \mu(F \setminus E) = \mu(F).$$

(b) We verify the requirements of an equivalence relation.
- $E \sim E$ since $\mu(E\triangle E) = \mu((E \setminus E) \cup (E \setminus E)) = \mu(\emptyset) = 0$.
- $E \sim F \implies F \sim E$ since $E\triangle F = F\triangle E$
- If $E \sim F$ and $F \sim G$, then $E \sim G$ since $E\triangle G \subset (E\triangle F) \cup (F\triangle G)$ and so
$$\mu(E\triangle G) \leq \mu(E\triangle F) + \mu(F\triangle G) = 0.$$

(c) Write it out:
$$\begin{aligned}\rho(E, G) &= \mu(E \setminus G) + \mu(G \setminus E)\\ &= \mu((E \cap F^c) \setminus G) + \mu((E \cap F) \setminus G) + \mu((G \cap F^c) \setminus E) + \mu((G \cap F) \setminus E)\\ &\leq \mu(E \setminus F) + \mu(F \setminus G) + \mu(G \setminus F) + \mu(F \setminus E)\\ &= \rho(E, F) + \rho(F, G).\end{aligned}$$

Folland 1.13. Let μ be a σ-finite measure on $(X, \mathcal{M})$, we seek to show that μ is semi-finite. Since μ is σ-finite there exists a disjoint sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}$ with $\bigcup_{n=1}^{\infty} A_n = X$ and $\mu(A_n) < \infty$. Let $E \in \mathcal{M}$ with $\mu(E) = \infty$. Let $E_n = A_n \cap E$, so that
$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} A_n \cap E = X \cap E = E.$$
In particular (note that E_n are disjoint since A_n are)
$$\infty = \mu(E) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$
We know that $\mu(E_n) \leq \mu(A_n) < \infty$ and that $\mu(E_n)$ cannot be zero for all n or else the sum wouldn't diverge. Thus there is some $E_m \subset E$ such that $0 < \mu(E_m) < \infty$ as required.

Folland 1.14. Suppose not. That is, suppose there is some $C > 0$ such that for all $F \subset E$, either $\mu(F) = \infty$ or $\mu(F) \leq C$. Since μ is semi-finite and $\mu(E) = \infty$ there exists $F_1 \subset E$ with $0 < \mu(F_1) < \infty$. If $\mu(F_1) > C$ we're done so assume that $\mu(F_1) \leq C$. Then $\mu(E \setminus F_1) = \infty$ and by semi-finiteness there exists $G \subset E \setminus F_1$ with $0 < \mu(G) < \infty$. Notice that F_1 and G are disjoint and so
$$\mu(F_1 \cup G) = \mu(F_1) + \mu(G) > \mu(F_1).$$
Furthermore $F_1 \cup G \subset E$ so $\mu(F_1 \cup G) \leq C$. Let $F_2 = F_1 \cup G$, and iteratively construct a strictly sequence
$$\mu(F_1) < \mu(F_2) < \mu(F_3) < \cdots < C.$$
This is a strictly increasing sequence bounded above so it converges to some limit $L \leq C$. Let $F = \bigcup_{i=1}^{\infty} F_i$, then by continuity from below
$$\mu(F) = \lim_{n\to\infty} \mu(F_n) = L \leq C < \infty.$$
In particular $\mu(E \setminus F) = \mu(E) - \mu(F) = \infty$. By semi-finiteness there exists $A \subset E \setminus F$ with $0 < \mu(A) < \infty$, but then $\mu(F \cup A) = L + \mu(A) > L$. But $\mu(F \cup A) < \infty$ and therefore $\mu(F \cup A) \leq C$ so $F \cup A$ would have been contained in our sequence, contradicting the fact that L was in fact the limit.

Folland 1.15. Let $\mu_0(E) = \sup\{\mu(F) : F \subset E \text{ and } \mu(F) < \infty\}$.

(a) We should first show that μ_0 is a measure. First of all the only subset of the empty set is the empty set itself $\mu_0(\emptyset) = \sup\{\mu(\emptyset)\} = 0$. Now let $\{E_i\}_{i\in\mathbb{N}}$ be disjoint and let $E = \bigcup_{i=1}^{\infty} E_i$. Our goal is to show

$$\mu_0(E) = \sum_{i=1}^{\infty} \mu_0(E_i).$$

Take any $F \subset E$ with $\mu(F) < \infty$ and set $F_i = F \cap E_i$. Then $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty}(F \cap E_i) = F \cap E = F$. Moreover (1) the F_i are disjoint and (2) $F_i \subset E_i$ with $\mu(F_i) \le \mu(F) < \infty$ so

$$\mu(F) = \sum_{i=1}^{\infty} \mu(F_i) \le \sum_{i=1}^{\infty} \mu_0(E_i).$$

Since $F \subset E$ with $\mu(F) < \infty$ was arbitrary we can take the supremum over all such F to obtain

$$\mu_0(E) \le \sum_{i=1}^{\infty} \mu_0(E_i).$$

On the other hand if $\mu_0(E) = \infty$ then $\mu_0(E) \ge \sum_{i=1}^{\infty} \mu_0(E_i)$ holds immediately. So assume $\mu_0(E) < \infty$. Notice that $\mu_0(E_i) \le \mu_0(E) < \infty$, so let $\varepsilon > 0$ and choose $F_i \subset E_i$ such that $\mu_0(E_i) - \varepsilon/2^i \le \mu_0(F_i)$. Then notice that $\bigcup_{i=1}^{n} F_i \subset E$ and has finite measure, moreover the F_i are disjoint. Thus

$$\mu_0(E) \ge \mu\left(\bigcup_{i=1}^{n} F_i\right) = \sum_{i=1}^{n} \mu(F_i) \ge \sum_{i=1}^{n} \mu_0(E_i) - \sum_{i=1}^{n} \frac{\varepsilon}{2^i}.$$

Letting $n \to \infty$ yields

$$\mu_0(E) \ge \sum_{i=1}^{\infty} \mu_0(E_i) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we conclude the desired result.

That μ_0 is semi-finite is immediate. Suppose $\mu_0(E) = \infty$, then by definition of the supremum for all $n \in \mathbb{N}$ there is $F \subset E$ with $\mu(F) < \infty$ such that $\mu(F) > n > 0$. In particular $\mu(F) = \mu_0(F)$ so μ_0 is semi-finite.

(b) If $\mu(E) < \infty$, then immediately $\mu(E) = \mu_0(E)$. If $\mu(E) = \infty$, then by Problem 1.14 for any $C > 0$, there is some $F \subset E$ with $\mu(F) < \infty$ such that $\mu(F) > C$. In particular

$$\mu_0(E) = \sup\{\mu(F) : F \subset E \text{ and } \mu(F) < \infty\} = \infty.$$

(c) Define

$$\nu(E) = \begin{cases} 0 & \text{if } E \text{ is } \sigma\text{-finite} \\ \infty & \text{if } E \text{ is not } \sigma\text{-finite} \end{cases},$$

where E being σ-finite means it can be written as the countable union of finite measure sets. Now we show that ν is a measure. $\nu(\emptyset) = 0$ since the empty set is σ-finite. Now let $E = \bigcup_{n=1}^{\infty} E_n$ be a disjoint union. If each E_n is σ-finite then so is E and so $\nu(E) = 0 = \sum_{n=1}^{\infty} \nu(E_n)$. If any one of the E_n is not σ-finite then neither is E. (If E was σ-finite then $E = \bigcup_{j=1}^{\infty} F_j$ where F_j have finite measure and then

$$E_n = E_n \cap E = E_n \cap \bigcup_{j=1}^{\infty} F_j = \bigcup_{j=1}^{\infty} (E_n \cap F_j)$$

and $\mu(E_n \cap F_j) < \infty$ so E_n is σ-finite, a contradiction). Hence $\nu(E) = \infty = \sum_{n=1}^{\infty} \nu(E_n)$.

It just remains to show that $\mu = \mu_0 + \nu$. If $\mu(E) < \infty$ or if E is not σ-finite then this is obvious. If $\mu(E) = \infty$ but E is σ-finite exercise 1.13 guarantees μ is semi-finite with respect to E and exercise 1.14 guarantees arbitrary large finite measure subsets of E. In particular $\mu_0(E) = \infty$. In summary:

$$\mu(E) = \begin{cases} \mu(E) + 0 & \mu(E) < \infty \\ \infty + 0 & \mu(E) = \infty \text{ and } E \text{ is } \sigma\text{-finite} \\ \mu_0(E) + \infty & \mu(E) = \infty \text{ and } E \text{ isn't } \sigma\text{-finite} \end{cases} = \mu_0(E) + \nu(E).$$

Folland 1.16.

(a) Let $X = \bigcup_{n=1}^{\infty}$ be a disjoint union with $\mu(E_n) < \infty$. Let $E \in \tilde{\mathcal{M}}$. Then by local measurability $E \cap E_n \in \mathcal{M}$ and so

$$E = E \cap X = E \cap \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap E_n) \in \mathcal{M}.$$

(b) $\tilde{\mathcal{M}}$ is non-empty since $\mathcal{M}$ is non-empty. $\tilde{\mathcal{M}}$ is closed under complement since if $E \in \tilde{\mathcal{M}}$ we have

$$E^c \cap A = (E^c \cup A^c) \cap A = (E \cap A)^c \cap A \in \mathcal{M}$$

for all $A \in \mathcal{M}$ with $\mu(A) < \infty$. $\tilde{\mathcal{M}}$ is closed under countable union since for $\{E_n\}_{n\in\mathbb{N}}$ in $\tilde{\mathcal{M}}$ we have

$$\left(\bigcup_{n=1}^{\infty} E_n\right) \cap A = \bigcup_{n=1}^{\infty} (E_n \cap A) \in \mathcal{M}$$

for all $A \in \mathcal{M}$ with $\mu(A) < \infty$.

(c) First we show that $\tilde{\mu}$ defined on $\tilde{\mathcal{M}}$ by

$$\tilde{\mu}(E) = \begin{cases} \mu(E) & E \in \mathcal{M} \\ \infty & \text{otherwise} \end{cases}$$

is a measure. Immediately $\tilde{\mu}(\emptyset) = \mu(\emptyset) = 0$. Now let $E = \bigcup_{n=1}^{\infty} E_n$ be a disjoint union in $\tilde{\mathcal{M}}$. If E is measurable with $\mu(E) < \infty$, then by local measurability of E_n we have $E_n = E_n \cap E \in \mathcal{M}$ and so

$$\tilde{\mu}(E) = \mu(E) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \tilde{\mu}(E_n).$$

If all the E_n are measurable then so is E, so assume that at least one of the E_n is not measurable. If E is measurable with $\mu(E) = \infty$ or if E is only locally measurable we have $\tilde{\mu}(E) = \infty = \sum_{n=1}^{\infty} \tilde{\mu}(E_n)$. Now we show that $\tilde{\mu}$ is saturated. Let $E \subset X$ be such that $E \cap A \in \tilde{\mathcal{M}}$ for all $A \in \tilde{\mathcal{M}}$ with $\tilde{\mu}(A) < \infty$. By definition of $\tilde{\mu}$ this is equivalent to $E \cap A \in \tilde{\mathcal{M}}$ for all $A \in \mathcal{M}$ with $\mu(A) < \infty$. Since $E \cap A \in \tilde{\mathcal{M}}$ we can use local measurability to intersect it with A to obtain $(E \cap A) \cap A = E \cap A \in \mathcal{M}$, or in other words $E \in \tilde{\mathcal{M}}$.

(d) Let $N \in \tilde{\mathcal{M}}$ with $\tilde{\mu}(N) = 0$. By definition of $\tilde{\mu}$ we have $N \in \mathcal{M}$ and $\mu(N) = 0$. By completeness of μ we have $Z \in \mathcal{M}$ for any $Z \subset N$. Finally since $\mathcal{M} \subset \tilde{\mathcal{M}}$ we have that $Z \in \tilde{\mathcal{M}}$ and so $\tilde{\mu}$ is complete.

(e) Notice that $\underline{\mu}$ is exactly the semi-finite part of $\tilde{\mu}$. Indeed

$$\underline{\mu}(E) = \sup\{\mu(A) : A \in \mathcal{M} \text{ and } A \subset E\} = \sup\{\tilde{\mu}(A) : A \subset E \text{ and } \tilde{\mu}(A) < \infty\}$$

This is clear if for all $A \subset E$ we have $\mu(A) < \infty$. But even if there is $A \subset E$ with $\mu(A) = \infty$, by semi-finiteness there exist $F \subset A \subset E$ of arbitrarily large finite measure. Hence the supremum will still agree. In particular as the semi-finite part of a measure we have that $\underline{\mu}$ is a measure. $\underline{\mu}$ extends μ since for $E \in \mathcal{M}$

$$\underline{\mu}(E) = \sup\{\mu(A) : A \in \mathcal{M} \text{ and } A \subset E\} = \mu(E).$$

Finally $\underline{\mu}$ is saturated since for any $E \subset X$ with $E \cap A \in \tilde{\mathcal{M}}$ for all $A \in \tilde{\mathcal{M}}$ with $\underline{\mu}(A) < \infty$ we can consider any $A \in \mathcal{M}$ and note that $\underline{\mu}(A) = \mu(A) < \infty$ and since $E \cap A \in \tilde{\mathcal{M}}$ we have $(E \cap A) \cap A = E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ with $\mu(A) < \infty$. In particular $E \in \tilde{\mathcal{M}}$.

(f) First μ is a measure since $\mu(\emptyset) = \mu_0(\emptyset \cap X_1) = \mu_0(\emptyset) = 0$ and for $E = \bigcup_{n=1}^{\infty} E_n$ disjoint in $\mathcal{M}$ we have

$$\mu(E) = \mu_0(E \cap X_1) = \mu_0\left(\bigcup_{n=1}^{\infty} (E_n \cap X_1)\right) = \sum_{n=1}^{\infty} \mu_0(E_n \cap X_1) = \sum_{n=1}^{\infty} \mu(E_n)$$

since $E_n \cap X_1$ are disjoint and μ_0 is a measure. Let $E \subset X$ and $A \in \mathcal{M}$ with $\mu(A) < \infty$. Then either A or A^c is countable. However, notice that $\mu(A) = \mu_0(A \cap X_1) < \infty$, so the portion of A residing X_1 is finite. But then $X_1 \setminus A$ is uncountable and so A^c cannot be countable. If A is countable so is $E \cap A$ and hence $E \cap A \in \mathcal{M}$. Now finally, consider $E = X_2$. We have $E^c = X_1$ so $E \notin \mathcal{M}$ since neither E nor E^c are countable and hence $\tilde{\mu}(E) = \infty$. But on the other hand all subsets of E are disjoint from X_1 so $\mu(A) = \mu_0(A \cap X_1) = \mu_0(\emptyset) = 0$ for any $A \subset E$, hence $\underline{\mu}(E) = 0$.

Folland 1.17. Let $A = \bigcup_{j=1}^{\infty} A_j$. Then by subadditivity we have

$$\mu^\star(E \cap A) = \mu^\star\left(\bigcup_{j=1}^{\infty}(E \cap A_j)\right) \leq \sum_{j=1}^{\infty} \mu^\star(E \cap A_j).$$

On the other hand, let $F_j = E \cap \bigcup_{k=j}^{\infty} A_k$. Then

$$\mu^\star(F_j) = \mu^\star(E \cap A_j) + \mu^\star(F_{j+1})$$

and $F_1 = E \cap A$, so recursively we obtain

$$\mu^\star(E \cap A) = \sum_{j=1}^{n} \mu^\star(E \cap A_j) + \mu^\star(F_{n+1}) \geq \sum_{j=1}^{n} \mu^\star(E \cap A_j)$$

for any $n \in \mathbb{N}$. Letting $n \to \infty$ yields the desired result.

Folland 1.18. First recall the definition

$$\mu^\star(E) = \inf\left\{\sum_{n=1}^{\infty} \mu_0(A_n) : E \subset \bigcup_{n=1}^{\infty} A_n \text{ and } A_n \in \mathcal{A}\right\}$$

(a) By the definition of infimum there exist $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that

$$\mu^\star(E) + \varepsilon \geq \sum_{n=1}^{\infty} \mu_0(A_n) = \sum_{n=1}^{\infty} \mu^\star(A_n) \geq \mu^\star\left(\bigcup_{n=1}^{\infty} A_n\right),$$

where the last inequality follows from subadditivity. Hence Let $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_\sigma$.

(b) Suppose E is $\mu^\star$-measurable. Let $A_n \in \mathcal{A}_\sigma$ be such that $\mu^\star(A_n) \leq \mu^\star(E) + 1/n$. Let $B = \bigcap_{n=1}^{\infty} A_n \in \mathcal{A}_{\sigma\delta}$. Then since $E \subset A_n$ for all $n \in \mathbb{N}$, we have $E \subset B$ also. Then since E is $\mu^\star$-measurable:

$$\mu^\star(E) + 1/n \geq \mu^\star(A_n) \geq \mu^\star(B) = \mu^\star(B \cap E) + \mu^\star(B \cap E^c) = \mu^\star(E) + \mu^\star(B \setminus E) \quad \forall n \in \mathbb{N}.$$

Then since $\mu^\star(E) < \infty$ we have $0 \leq \mu^\star(B \setminus E) \leq 1/n$ for all $n \in \mathbb{N}$, and so $\mu^\star(B \setminus E) = 0$. On the other hand suppose there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^\star(B \setminus A) = 0$. It suffices to show that

$$\mu^\star(F) \geq \mu^\star(F \cap E) + \mu^\star(F \cap E^c)$$

for all $F \subset X$ with $\mu^\star(F) < \infty$. Since $B \in \mathcal{A}_{\sigma\delta}$ it is $\mu^\star$ measurable and so

$$\mu^\star(F) = \mu^\star(F \cap B) + \mu^\star(F \cap B^c) \geq \mu^\star(F \cap E) + \mu^\star(F \cap B^c).$$

Moreover, since $E \subset B$, we can write $E^c = B^c \cup (B \cap E^c)$ so

$$\mu^\star(F \cap E^c) = \mu^\star(F \cap (B^c \cup (B \cap E^c))) \leq \mu^\star(F \cap B^c) + \underbrace{\mu^\star(F \cap (B \cap E^c))}_{\leq \mu^\star(B \cap E^c) = 0} \leq \mu^\star(F \cap B^c).$$

So we see that

$$\mu^\star(F) \geq \mu^\star(F \cap E) + \mu^\star(F \cap E^c)$$

as desired.

(c) If μ_0 is σ-finite, write $X = \bigcup_{j=1}^{\infty} X_j$ where each X_j has finite μ_0 measure. Let $E_j = E \cap X_j$ which has finite $\mu^\star$ measure and $E = \bigcup_{j=1}^{\infty} E_j$. Fix $n \in \mathbb{N}$ and choose $C_j \supset E$ so that

$$\mu^\star(E_j) + \frac{1}{n2^j} \geq \mu^\star(C_j) = \mu^\star(E_j) + \mu^\star(C_j \setminus E_j) \implies \mu^\star(C_j \setminus E_j) \leq \frac{1}{n2^j}.$$

Let $B_n = \bigcup_{j=1}^{\infty} C_j \in \mathcal{A}_\sigma$ and notice that $E^c \subset E_j^c$. Now

$$\mu^\star(B_n \setminus E) = \mu^\star\left(\bigcup_{j=1}^{\infty}(C_j \cap E^c)\right) \leq \mu^\star\left(\bigcup_{j=1}^{\infty}(C_j \cap E_j^c)\right) \leq \sum_{j=1}^{\infty} \mu^\star(C_j \setminus E_j) \leq \frac{1}{n}.$$

Now let $B = \bigcap_{n=1}^{\infty} B_n \in \mathcal{A}_{\sigma\delta}$ so that $\mu^\star(B \setminus E) \leq \mu^\star(B_n \setminus E) \leq 1/n$ for all $n \in \mathbb{N}$. In particular $\mu^\star(B \setminus E) = 0$. For the other direction we did not need $\mu^\star(E) < \infty$.

Folland 1.19. If E is $\mu^\star$-measurable then

$$\mu^\star(X) = \mu^\star(E) + \mu^\star(E^c) \implies \mu^\star(E) = \mu^\star(X) - \mu^\star(E^c) = \mu_0(X) - \mu^\star(E^c) = \mu_\star(E).$$

On the other hand if $\mu^\star(E) = \mu_\star(E)$ choose $A_n \in \mathcal{A}_\sigma$ so that $E \subset A_n$ and $\mu^\star(A_n) \leq \mu^\star(E) + 1/n$. Notice that A_n is $\mu^\star$-measurable, so

$$\mu^\star(E^c) = \mu^\star(E^c \cap A_n) + \mu^\star(E^c \cap A_n^c) = \mu^\star(A_n \setminus E) + \mu^\star(A_n^c).$$

Moreover since $\mu^\star(E) = \mu_\star(E)$ and A_n is $\mu^\star$-measurable:

$$\mu^\star(E) = \mu_\star(E) = \mu^\star(X) - \mu^\star(E^c) = \mu^\star(A_n) + \mu^\star(A_n^c) - \mu^\star(E^c).$$

Taken together we have

$$\mu^\star(A_n \setminus E) = \mu^\star(E^c) - \mu^\star(A_n^c) = \mu^\star(A_n) - \mu^\star(E) \leq \frac{1}{n}.$$

Now let $B = \bigcap_{n=1}^\infty A_n \in \mathcal{A}_{\sigma\delta}$ so that

$$\mu^\star(A \setminus E) \leq \mu^\star(A_n \setminus E) \leq \frac{1}{n}$$

for all $n \in \mathbb{N}$. It follows that $\mu^\star(B \setminus E) = 0$ for some $B \in \mathcal{A}_{\sigma\delta}$ and so by 1.18 (b) we have that E is $\mu^\star$-measurable.

Folland 1.20. First write the definition

$$\mu^+(E) = \inf\left\{\sum_{n=1}^\infty \overline{\mu}(A_n) : \bigcup_{n=1}^\infty A_n \text{ and } A_n \in \mathcal{M}^\star\right\}.$$

(a) Let $\varepsilon > 0$. By infimum properties there exists $A = \bigcup_{n=1}^\infty A_n \supset E$ such that

$$\mu^+(E) + \varepsilon \geq \sum_{n=1}^\infty \overline{\mu}(A_n) = \sum_{n=1}^\infty \mu^\star(A_n) \geq \mu^\star(A) \geq \mu^\star(E).$$

Moreover, if there is an $A \in \mathcal{M}^\star$ with $A \supset E$ and $\mu^\star(A) = \mu^\star(E)$ then

$$\mu^+(E) \leq \overline{\mu}(A) = \mu^\star(A) = \mu^\star(E).$$

And if $\mu^+(E) = \mu^\star(E)$ then by properties of infimum there exists $\{A_j\}_{j=1}^\infty$ such that

$$\sum_{j=1}^\infty \overline{\mu}(A_j) \leq \mu^+(E) + \frac{1}{n}.$$

Now set $A_n = \bigcup_{j=1}^\infty A_j$ so that

$$\mu^\star(A_n) \leq \sum_{j=1}^\infty \mu^\star(A_j) = \sum_{j=1}^\infty \overline{\mu}(A_j) \leq \mu^+(E) + \frac{1}{n}.$$

Finally let $A = \bigcap_{n=1}^\infty A_n$ so that

$$\mu^\star(A) \leq \mu^\star(A_n) \leq \mu^+(E) + \frac{1}{n},$$

for all $n \in \mathbb{N}$. It follows that $\mu^\star(A) \leq \mu^+(E) = \mu^*(E)$. Moreover, $E \subset A$ and so $\mu^\star(E) \leq \mu^\star(A)$ and therefore we have $\mu^\star(E) = \mu^\star(A)$.

(b) Let $E \subset X$, we already have $\mu^\star(E) \leq \mu^+(E)$. Let $\varepsilon > 0$ then by 1.18 (a) there is $A \in \mathcal{A}_\sigma$ such that $A \supset E$, $A \in \mathcal{A}_\sigma \subset \mathcal{M}^\star$ and $\mu^\star(A) \leq \mu^\star(E) + \varepsilon$. Hence

$$\mu^\star(E) \geq \mu^\star(A) - \varepsilon = \overline{\mu}(A) - \varepsilon \geq \mu^+(E) - \varepsilon.$$

(c) $\mathcal{P}(X) = \{\emptyset, \{0\}, \{1\}, X\}$. Let $\mu^\star(\emptyset) = 0$, $\mu^\star(\{0\}) = \mu^\star(\{1\}) = 2$ and $\mu^\star(X) = 3$. This defines a valid outer measure. But then $\{0\}$ and $\{1\}$ are not $\mu^\star$-measurable. Hence $\mu^+(\{0\}) = \mu^\star(X) = 4 \neq 2 = \mu^\star(\{0\})$.

Folland 1.21. Let $\mathcal{M}$ be the $\mu^\star$-measurable sets. Let $E \in \tilde{\mathcal{M}}$, namely $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ with $\overline{\mu}(A) < \infty$. We must show that E is $\mu^\star$-measurable. Let $F \subset X$ with $\mu^\star(F) < \infty$. Let $\varepsilon > 0$. Then by 1.18 (a) there exists $A \supset F$ with $\mu^\star(A) \leq \mu^\star(F) + \varepsilon$. Moreover, A is $\mu^\star$-measurable with finite outer measure and since $E \in \tilde{\mathcal{M}}$ we have $E \cap A$ is $\mu^\star$-measurable. Thus

$$\begin{aligned}\mu^\star(F) + \varepsilon &\geq \mu^\star(A) \\ &= \mu^\star(A \cap (E \cap A)) + \mu^\star(A \cap (E \cap A)^c) \\ &= \mu^\star(A \cap E) + \mu^\star(A \cap (E^c \cup A^c)) \\ &= \mu^\star(A \cap E) + \mu^\star(A \cap E^c) \\ &\geq \mu^\star(F \cap E) + \mu^\star(F \cap E^c)\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we see that $\mu^\star(F) \geq \mu^\star(F \cap E) + \mu^\star(F \cap E^c)$ and so E is $\mu^\star$-measurable as desired.

Folland 1.22.

Folland 1.23.

Folland 1.24.

(a) First notice that

$$(A \setminus B) \cap E = (A \cap B^c) \cap E = (A \cap E) \cap B^c = (B \cap E) \cap B^c = \emptyset.$$

And similarly $(B \setminus A) \cap E = \emptyset$. Hence $A \triangle B \subset E^c$. In particular

$$\mu^\star(X) = \mu^\star(E) \leq \mu^\star((A \triangle B)^c) \leq \mu^\star(X),$$

from which it follows that $\mu^\star((A \triangle B)^c) = \mu^\star(X)$. Moreover, since $A \triangle B$ is $\mu^\star$-measurable, we write

$$\mu^\star(X) = \mu^\star(A \triangle B) + \mu^\star((A \triangle B)^c) = \mu^\star(A \triangle B) + \mu^\star(X).$$

So $\mu(X) = \mu(A \triangle B) + \mu(X)$ and therefore $\mu(A \triangle B) = 0$. Then by problem 1.12 (a) we have that $\mu(A) = \mu(B)$.

(b) The collection $\mathcal{M}_E = \{A \cap E : A \in \mathcal{M}\}$ is a σ-algebra on E. It is non-empty since $\mathcal{M}$ is non-empty. The complement of $A \cap E$ in E is $A^c \cap E$. Finally $\bigcup_{n=1}^\infty (A_n \cap E) = (\bigcup_{n=1}^\infty A_n) \cap E$. Now to show that $\nu(A \cap E) = \mu(A)$ is a measure. First $\nu(\emptyset) = \nu(\emptyset \cap E) = \mu(\emptyset) = 0$. Now for a disjoint collection $\{A_n \cap E\}_{n \in \mathbb{N}}$ of sets of in $\mathcal{M}_E$ we may not have the $\{A_n\}_{n \in \mathbb{N}}$ be disjoint in $\mathcal{M}$, so let $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$ which are disjoint and $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n$. Moreover

$$B_n \cap E = (A_n \cap E) \setminus \bigcup_{k=1}^{n-1} (A_k \cap E) = A_n \cap E$$

since the $A_k \cap E$ *are* disjoint. And so finally

$$\nu\left(\bigcup_{n=1}^\infty (A_n \cap E)\right) = \nu\left(\bigcup_{n=1}^\infty B_n \cap E\right) = \mu\left(\bigcup_{n=1}^\infty B_n\right) = \sum_{n=1}^\infty \mu(B_n) = \sum_{n=1}^\infty \nu(B_n \cap E) = \sum_{n=1}^\infty \nu(A_n \cap E).$$

Folland 1.25. Recall Theorem 1.19: If $E \subset \mathbb{R}$ the following are equivalent.

(a) $E \in \mathcal{M}_\mu$
(b) $E = V \setminus N$ where V is a G_δ set and $\mu(N) = 0$
(c) $E = H \cup N$ where H is an F_σ and $\mu(N) = 0$.

Folland already proves (a) $\Longrightarrow$ (b) and (a) $\Longrightarrow$ (c) for finite $\mu(E)$. Moreover, (a) $\Longrightarrow$ (c) implies (a) $\Longrightarrow$ (b) since if $E \in \mathcal{M}_\mu$ so $E^c \in \mathcal{M}_\mu$. Then $E^c = H \cup N$. Let $V = H^c$ which is a G_δ set since $V = \left(\bigcup_{n=1}^\infty H_n\right)^c = \bigcap_{n=1}^\infty H_n^c = \bigcap_{n=1}^\infty V_n$, and each V_n is open since each H_n is closed. And $E = V \setminus N$ since $E = (H \cup N)^c = V \cap N^c = V \setminus N$. Now we need just show that (a) $\Longrightarrow$ (c) for the case when $\mu(E) = \infty$. Let $E_j = E \cap (j, j+1]$ for $j \in \mathbb{Z}$. Then $\mu(E_j) < \infty$ and so there exists an F_σ set H_j and a null N_j such that $E_j = H_j \cup N_j$. Finally

$$E = \bigcup_{j \in \mathbb{Z}} E_j = \bigcup_{j \in \mathbb{Z}} (H_j \cup N_j) = H \cup N,$$

where $H = \bigcup_{j \in \mathbb{Z}} H_j$ and $N = \bigcup_{j \in \mathbb{Z}}$. Notice that H is still F_σ, and that $\mu(N) \leq \mu(N_j) = 0$.

Folland 1.26. Recall proposition 1.20: If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\varepsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \varepsilon$. Recall also $\mu(E) = \inf\left\{\sum_{j=1}^\infty \mu((a_j, b_j)) : E \subset \bigcup_{j=1}^\infty (a_j, b_j)\right\}$. Let $\varepsilon > 0$ and denote $I_j = (a_j, b_j)$. By definition of the infimum there exist $\{I_j\}_{j\in\mathbb{N}}$ such that

$$\sum_{j=1}^\infty \mu(I_j) \le \mu(E) + \varepsilon/2,$$

and since $\mu(E) < \infty$ this sum is convergent. So there is $N \in \mathbb{N}$ such that $\sum_{j=N}^\infty \mu(I_j) < \varepsilon/2$. Let $A = \bigcup_{j=1}^{N-1} I_j$. Then

$$\mu(A \setminus E) \le \mu(I \setminus E) \le \frac{\varepsilon}{2} \qquad \text{and} \qquad \mu(E \setminus A) \le \mu(I \setminus A) = \mu\left(\bigcup_{j=N}^\infty I_j\right) < \frac{\varepsilon}{2}.$$

Taken together we have $\mu(E \triangle A) = \mu(E \setminus A) + \mu(A \setminus E) < \varepsilon$.

Folland 1.27.

Folland 1.28.

Folland 1.29.

(a)

Folland 1.30. Suppose not, namely that there is $\alpha < 1$ such that for all open intervals I we have $m(E \cap I) \le \alpha m(I)$. Assuming $m(E) < \infty$. By the definition of m there is a collection of intervals such that $I = \bigcup_{j=1}^\infty I_j \supset E$ and

$$\sum_{j=1}^\infty \mu(I_j) < (1+\varepsilon) m(E).$$

But then

$$m(E) = m(E \cap I) \le \sum_{j=1}^\infty m(E \cap I_j) \le \alpha \sum_{j=1}^\infty m(I_j) < \alpha(1+\varepsilon) m(E).$$

This is a contradiction if $\alpha(1+\varepsilon) < 1$ or equivalently if $\varepsilon < 1/\alpha - 1$. Such an $\varepsilon > 0$ can be chosen since $\alpha < 1$ implies $1/\alpha - 1 > 0$. If $m(E) = \infty$ we use σ-finiteness to write $E = \bigcup_{n=1}^\infty E_n$ each with $\mu(E_n) < \infty$. At least one must be positive from which for any $\alpha > 1$ there exists an open interval such that $m(E_k \cap I) > \alpha m(I)$. Finally $m(E \cap I) \ge m(E_k \cap I) > \alpha m(I)$ as desired.

Folland 1.31.

Folland 1.32.

Folland 1.33.

2. Integration

Folland 2.1.

Folland 3.1. Let ν be a signed measure on $(X, \mathcal{M})$. Let $E_1 \subset E_2 \subset \cdots$, and write $\bigcup_{j=1}^\infty E_j = E_1 \cup \Big[\bigcup_{j=2}^\infty (E_j \setminus E_{j-1})\Big]$. Then

$$\nu\left(\bigcup_{j=1}^\infty E_j\right) = \nu(E_1) + \sum_{j=1}^\infty \nu(E_j \setminus E_{j-1}) = \nu(E_1) + \lim_{n\to\infty}\sum_{j=1}^n \nu(E_j \setminus E_{j-1}) = \lim_{n\to\infty} \nu(E_n).$$

Now suppose $E_1 \supset E_2 \supset \cdots$ with $\nu(E_1)$ finite. Write $F_j = E_1 \setminus E_j$ so that $F_1 \subset F_2 \subset \cdots$. Now

$$\nu\left(\bigcup_{j=1}^\infty F_j\right) = \nu\left(\bigcup_{j=1}^\infty (E_1 \cap E_j^c)\right) = \nu\left(E_1 \cap \left(\bigcap_{j=1}^\infty E_j\right)^c\right) = \nu(E_1) - \nu\left(\bigcap_{j=1}^\infty E_j\right).$$

Moreover $\nu(F_j) = \nu(E_1) - \nu(E_j)$. Finally by continuity from below:

$$\nu(E_1) - \lim_{j\to\infty}\nu(E_j) = \lim_{j\to\infty}\nu(F_j) = \nu\left(\bigcup_{j=1}^\infty F_j\right) = \nu(E_1) - \nu\left(\bigcap_{j=1}^\infty E_j\right),$$

and since $\nu(E_1)$ is finite, the claim follows.

Folland 3.2. Let ν be a signed measure. Let $\nu = \nu^+ - \nu^-$ be its Jordan decomposition and $X = P \cup N$ be the associated Hahn decomposition. Namely $\nu^+(E) = \nu(E \cap P)$, $\nu^-(E) = -\nu(E \cap N)$, and $N \cap P = \emptyset$.

If E is ν-null, then $|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu(E \cap N) = 0 - 0 = 0$. Since $E \cap P, E \cap N \subset E$. Conversely, if $|\nu|(E) = 0$, then for any $A \subset E$: $|\nu|(A) \le |\nu|(E) = 0$. In particular $\nu^+(A) + \nu^-(A) = 0$ implying $\nu^+(A) = \nu^-(A) = 0$. So $\nu(A) = 0$ and E is ν-null.

Suppose $\nu \perp \mu$. Namely there is $X = A \cup B$ such that A is ν-null, B is μ-null, and $A \cap B = \emptyset$. Then by the above A is $|\nu|$-null also (since $|\nu|$ is a positive measure). So $|\nu| \perp \mu$.

Now suppose $|\nu| \perp \mu$. Namely there is $X = A \cup B$ such that A is $|\nu|$-null, B is μ-null, and $A \cap B = \emptyset$. Since $|\nu|(E) = 0 \implies v^\pm(E) = 0$, we have that A is also $\nu^\pm$-null, so $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Finally suppose that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Namely there are $X = A^\pm \cup B^\pm$ such that $A^\pm$ is $\nu^\pm$-null, $B^\pm$ is μ-null, and $A^\pm \cap B^\pm = \emptyset$. Then $A := A^+ \cap A^-$ is both ν^+ and ν^--null, hence it is ν-null. Moreover, $B := A^c = {A^+}^c \cup {A^-}^c = B^+ \cup B^-$ is μ-null and $A \cap B = \emptyset$. So $\nu \perp \mu$.

Folland 3.3. Let ν be a signed measure on $(X, \mathcal{M})$. First a lemma. For positive measures μ_1, μ_2 and a non-negative measurable function f

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2.$$

Indeed, first we prove the result for simple functions $s = \sum_{i=1}^n a_i \chi_{E_i}$ (w.l.o.g. assume E_i are disjoint). Then

$$\int s d(\mu_1 + \mu_2) = \sum_{i=1}^n a_i(\mu_1(E_i) + \mu_2(E_i)) = \sum_{i=1}^n a_i \mu_1(E_i) + \sum_{i=1}^n a_i \mu_2(E_i) = \int s d\mu_1 + \int s d\mu_2.$$

For any positive measurable f, there is a sequence of positive simple functions increasing monotonically to f, hence by the M.C.T.

$$\int f(d\mu_1 + \mu_2) = \lim_{n\to\infty}\int s_n d(\mu_1 + \mu_2) = \lim_{n\to\infty}\left(\int s_n d\mu_1 + \int s_n d\mu_2\right) = \int f d\mu_1 + \int f d\mu_2.$$

(a) By definition

$$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-) = \left\{\text{measurable } f : \int |f| d\nu^+ < \infty \text{ and } \int |f| d\nu^- < \infty\right\}.$$

If $f \in L^1(\nu)$, then

$$\int |f| d|\nu| = \int |f| d(\nu^+ + \nu^-) = \int |f| d\nu^+ + \int |f| d\nu^- < \infty.$$

On the other hand if $f \in L^1(|\nu|)$, then $\int |f| d\nu^\pm \le \int |f| d|\nu| < \infty$. So $L^1(\nu) \subset L^1(|\nu|)$ and $L^1(|\nu|) \subset L^1(\nu)$.

(b)

$$\left|\int f d\nu\right| = \left|\int f d\nu^+ - \int f d\nu^-\right| \leq \left|\int f d\nu^+\right| + \left|\int f d\nu^-\right| \leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|.$$

(c) Let P, N be the Hahn decomposition such that $\nu^+(A) = \nu(A \cap P)$ and $\nu^-(A) = -\nu(A \cap N)$. Then $|\chi_P - \chi_N| \leq 1$ and $\int_E \chi_P - \chi_N d\nu = \nu(E \cap P) - \nu(E \cap N) = |\nu|(E)$ so $\sup\{|\int_E f d\nu| : |f| \leq 1\} \geq |\nu|(E)$. On the other hand, for any measurable f with $|f| \leq 1$:

$$\left|\int_E f d\nu\right| = \left|\int f \chi_E d\nu\right| \leq \int |f \chi_E| d|\nu| \leq \int \chi_E d|\nu = |\nu|(E).$$

So $\sup\{|\int_E f d\nu| : |f| \leq 1\} \leq |\nu|(E)$ as desired.

Folland 3.4. Let P, N be the Hahn decomposition such that $\nu^+(A) = \nu(A \cap P)$ and $\nu^-(A) = -\nu(A \cap N)$. Then

$$\lambda(A) \geq \lambda(A \cap P) = \nu(A \cap P) + \mu(A \cap P) \geq \nu(A \cap P) = \nu^+(A)$$
$$\mu(A) \geq \mu(A \cap N) = \lambda(A \cap N) - \nu(A \cap N) \geq -\nu(A \cap N) = \nu^-(A).$$

Folland 3.5. $\nu_1 + \nu_2 = \nu_1^+ - \nu_1^- + \nu_2^+ - \nu_2^-$ is a signed measure, and $\nu_1^+ + \nu_2^+$ and $\nu_1^- + \nu_2^-$ are two positive measures satisfying the conditions of 3.4, hence: $\nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$ and $\nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$. Altogether

$$|\nu_1 + \nu_2| = (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \leq \nu_1^+ + \nu_2^+ + \nu_1^- + \nu_2^- = |\nu_1| + |\nu_2|.$$

Folland 3.6. For $\nu(E) = \int_E f d\mu$ we have: $P = \{x \in X : f(x) \geq 0\}$, $N = \{x \in X : f(x) < 0\}$, $\nu^+(E) = \int_E f^+ d\mu$, $\nu^-(E) = -\int_E f^- d\mu$, and $|\nu|(E) = \int_E |f| d\mu$. Here $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

Folland 3.7.

(a) First of all: $\nu^+(E) = \nu(E \cap P) \leq \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$. On the other hand, if $F \in \mathcal{M}$ with $F \subset E$, then $\nu(F) = \nu^+(F) - \nu^-(F) \leq \nu^+(F) \leq \nu^+(E)$. So $\nu^+(E) \geq \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$. Similarly, $\nu^-(E) = -\nu(E \cap N) \leq -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$. And on the other hand, if $F \in \mathcal{M}$ with $F \subset E$, then $\nu(F) = \nu^+(F) - \nu^-(F) \geq -\nu^-(F) \geq -\nu^-(E)$. So $\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\} \geq -\nu^-(E)$ or equivalently, $-\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\} \leq \nu^-(E)$.

(b) Since $|\nu(E \cap P)| + |\nu(E \cap N)| = \nu^+(E) + \nu^-(E) = |\nu|(E)$ and $(E \cap P) \cup (E \cap N) = E$, we have that $|\nu|(E) \leq \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n$ and disjoint, and $\bigcup_{j=1}^n E_j = E\}$. For the other direction, let $E_1, \ldots, E_n$ be disjoint with $\bigcup_{j=1}^n E_j = E$. Then

$$\sum_{j=1}^n |\nu(E_j)| \leq \sum_{j=1}^n |\nu|(E_j) = |\nu|(E).$$

Since $|\nu(A)| = |\nu^+(A) - \nu^-(A)| \leq \nu^+(A) + \nu^-(A) = |\nu|(A)$. In any case, we see that the supremum over all such $E_1, \ldots, E_n$ is bounded b $|\nu|(E)$.

Folland 3.8. Let $\nu = \nu^+ - \nu^-$ be a Jordan decomposition and $X = P \cup N$ be the associated Hahn decomposition. Namely $\nu^+(E) = \nu(E \cap P)$, $\nu^-(E) = -\nu(E \cap N)$, and $N \cap P = \emptyset$. (I also assume μ is a positive measure).

Let $\nu \ll \mu$ and suppose $\mu(E) = 0$. Then $\mu(E \cap P) \leq \mu(E) = 0$, so $\nu^+(E) = \nu(E \cap P) = 0$. Similarly, $\mu(E \cap N) = 0$ and so $\nu^-(E) = -\nu(E \cap N) = 0$. Namely $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$.

Now let $|\nu| \ll \mu$ and suppose $\mu(E) = 0$. Then $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0 \implies \nu^+(E) = \nu^-(E) = 0$.

Finally if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, and $\mu(E) = 0$, then $\nu(E) = \nu^+(E) - \nu^-(E) = 0 - 0 = 0$.

Folland 3.9. First suppose that $\nu_j \perp \mu$ for all $j \in \mathbb{N}$. Namely there exist disjoint sets M_j, N_j such that M_j is ν_j-null, N_j is μ-null and $M_j \cup N_j = X$ for all $j \in \mathbb{N}$. Consider, $N = \bigcap_{j=1}^\infty N_j$ which is ν_j-null for all j and so is $\sum_j \nu_j$-null. Moreover $M = N^c = \bigcup_{j=1}^\infty M_j$ is μ-null. Hence $\sum_j \nu_j \perp \mu$.

Now suppose $\nu_j \ll \mu$ for all $j \in \mathbb{N}$. If $\mu(E) = 0$, then $\sum_1^\infty \nu_j(E) = \sum_1^\infty 0 = 0$, so $\sum_j \nu_j \ll \mu$.

Folland 3.10. Let ν be the counting measure and $\mu = \sum_{n \in E} 2^{-n}$ on $\mathbb{N}$. Then $\mu(E) = 0 \implies E = \emptyset$ so $\nu \ll \mu$. However, taking $\varepsilon = 1/2$, there is no $\delta > 0$ such that $|\nu(E)| < 1/2$ whenever $\mu(E) < \delta$. Indeed for any $\delta > 0$, one can choose $n \in \mathbb{N}$ so that $\mu(\{n\}) = 2^{-n} < \delta$ but $|\nu(\{n\})| = 1 > 1/2$.

Folland 3.11.

(a) Let $\varepsilon > 0$. For any function in $L^1(\mu)$ we have $\left|\int_E f \,\mathrm{d}\mu\right| \leq \int_E |f| \,\mathrm{d}\mu =: \nu(E)$, here $\nu \ll \mu$ by construction. Moreover ν is finite since f is integrable, and so by Theorem 3.5 there is $\delta > 0$ such that $\left|\int_E f \,\mathrm{d}\mu\right| < \varepsilon$ whenever $\mu(E) < \delta$. For any finite collection $\{f_\alpha\}_{\alpha\in A}$, there is $\delta_\alpha > 0$ such that $\left|\int_E f \,\mathrm{d}\mu\right| < \varepsilon$ whenever $\mu(E) < \delta_\alpha$. Take $\delta = \min\{\delta_\alpha : \alpha \in A\}$, which exists since A is finite. Hence $\{f_\alpha\}_{\alpha\in A}$ is uniformly integrable.

(b) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $L^1(\mu)$ such that $f_n \xrightarrow{L^1} f$. Namely $\int_X |f_n - f| \,\mathrm{d}\mu \to 0$ as $n \to \infty$. Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\int |f_n - f| \,\mathrm{d}\mu < \varepsilon/2$ for $n > N$. By part (a) $\mathcal{F} := \{f\} \cup \{f_n\}_{n=1}^N$ is uniformly integrable. Namely there is $\delta > 0$ such that $\left|\int_E f \,\mathrm{d}\mu\right| < \varepsilon/2$ whenever $\mu(E) < \delta$ for each $f \in \mathcal{F}$. For $n > N$,

$$\left|\int_E f_n \,\mathrm{d}\mu\right| = \left|\int_E (f_n - f) \,\mathrm{d}\mu + \int_E f \,\mathrm{d}\mu\right| \leq \left|\int_E (f_n - f) \,\mathrm{d}\mu\right| + \left|\int_E f \,\mathrm{d}\mu\right| \leq \int_E |f_n - f| \,\mathrm{d}\mu + \left|\int_E f \,\mathrm{d}\mu\right|.$$

The first term is bounded by $\varepsilon/2$ and the second term is bounded by $\varepsilon/2$ so long as $\mu(E) < \delta$. Hence $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable (In fact $\{f\} \cup \{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable).

Folland 3.12. Let $E = A \times B$. Since ν_j, μ_j are σ-finite, and$(\nu_1 \times \nu_2) \ll (\mu_1 \times \mu_2)$, $\nu_j \ll \mu_j$, we have that all relevant Radon-Nikodym densities exist. Now by properties of the density and Fubini-Tonelli:

$$(\nu_1 \times \nu_2)(E) = \int_E \mathrm{d}(\nu_1 \times \nu_2) = \int_A \left(\int_B \mathrm{d}\nu_2\right) \mathrm{d}\nu_1 = \int_A \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1} \left(\int_B \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2} \,\mathrm{d}\mu_2\right) \mathrm{d}\mu_1 = \int_A \left(\int_B \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2} \,\mathrm{d}\mu_2\right) \mathrm{d}\mu_1.$$

One last appeal to Fubini-Tonelli yields

$$(\nu_1 \times \nu_2)(E) = \int_E \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2} \,\mathrm{d}(\mu_1 \times \mu_2),$$

and we conclude by uniqueness.

Folland 3.13.

(a) Let $X = [0, 1]$ equipped with the Borel σ-algebra. Let m be the Lebesgue measure and μ the counting measure. If $\mu(A) = 0$, then $A = \emptyset$ and $m(\emptyset) = 0$. So $m \ll \mu$. However $\mathrm{d}m \neq f \,\mathrm{d}\mu$ for any f. If $m(E) = \int_E f \,\mathrm{d}\mu$, then $0 = m(\{x\}) = \int_{\{x\}} f \,\mathrm{d}\mu = f(x)\mu(\{x\}) = f(x)$. Namely $f(x) = 0$ for all $x \in X$. But then $1 = m(X) = \int_X f \,\mathrm{d}\mu = \int_X 0 \,\mathrm{d}\mu = 0$, a contradiction.

(b) By way of contradiction suppose $\mu = \lambda + \rho$ is a Lebesgue decomposition such that $m \perp \lambda$ and $\rho \ll m$. Since $\rho \ll m$, $\rho(\{x\}) = 0$. But then $\lambda(\{x\}) = \mu(\{x\}) - \rho(\{x\}) = 1 - 0 = 1$. Now since $\lambda \perp m$, there is $N \subset [0, 1]$ such that N is m-null and N^c is λ-null. But for $x \in N^c$ we have $\{x\} \subset N^c$ and since N^c is λ-null we have $\lambda(\{x\}) = 0$. This contradicts the above unless $N^c = \emptyset$. But if $N^c = \emptyset$, then $N = [0, 1]$. But then clearly $N = [0, 1]$ is not m-null since $m([0, 1]) = 1$.

Folland 3.14. Let ν be a signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. We seek to show that if $\nu \ll \mu$, then there exists an extended μ-integrable $f : X \to [-\infty, \infty]$ such that $\mathrm{d}\nu = f \,\mathrm{d}\mu$.

(a) First it suffices to assume that μ is finite and ν is positive. Indeed once we prove the result for this case we can extend to σ-finite μ by writing $X = \bigcup_{j=1}^\infty E_j$ as a disjoint union with $\mu(E_j) < \infty$. Write $\mu_j(A) = \mu(E_j \cap A)$, so that $\mu = \sum_{j=1}^\infty \mu_j$. Note that each μ_j is a finite measure so there is an extended μ_j-integrable f_j such that $\nu_j(A) = \int_A f_j \,\mathrm{d}\mu_j$. Then take $f = f_j\chi_{E_j}$ and use $\nu = \sum_j \nu_j$. Then we can extend the result to signed ν by finding $f^\pm$ such that $\nu^+(A) = \int_A f^+ \,\mathrm{d}\mu$ and $\nu^-(A) = \int_A f^- \,\mathrm{d}\mu$. Finally take $f = f^+ - f^-$ so that

$$\nu(A) = \nu^+(A) - \nu^-(A) = \int_A f^+ - f^- \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu.$$

(b) Under these assumptions we show that there exists an $E \in \mathcal{M}$ that is σ-finite for ν such that $\mu(E) \geq \mu(F)$ for all F that are σ-finite for ν. Let $\mathcal{S} = \{S \in \mathcal{M} : S \text{ is } \sigma\text{-finite for } \nu\}$. $\mathcal{S}$ is non-empty since it contains $\emptyset$. Define $M := \sup\{\mu(S) : S \in \mathcal{S}\}$. By definition of supremum, there is a sequence of sets $S_1 \subset S_2 \subset \cdots$ such that $\mu(S_n)$ increases to M. Take $E = \bigcup_{n=1}^\infty S_n \in \mathcal{S}$ as a countable union of σ-finite sets. Now by continuity from below $\mu(E) = \lim_{n\to\infty} \mu(S_n) = M$. So the supremum is attained and hence $\mu(E) \geq \mu(F)$ for any $F \in \mathcal{S}$.

(c) Now restrict the measure $\nu_E(A) = \nu(A \cap E)$. By Radon-Nikodym, there exists $g : E \to [0, \infty)$ such that

$$\nu_E(A) = \int_{A \cap E} g \, \mathrm{d}\mu.$$

Moreover, notice that if $F \cap E = \emptyset$, then either $\mu(F) = \nu(F) = 0$ or $\mu(F) > 0$ and $\nu(F) = \infty$. Indeed if $\nu(F) < \infty$, then $F \in \mathcal{S}$ and therefore $\mu(E \cup F) > \mu(E)$ contradicting the maximality of E unless $\mu(F) = 0$. So if $\nu(F) < \infty$, then $\mu(F) = 0$ and $\nu \ll \mu \implies \nu(F) = 0$. On the other hand: $\mu(A) > 0 \implies \nu(A) = \infty$. Define $f : X \to [0, \infty]$ as

$$f(x) = \begin{cases} g(x) & x \in E \\ \infty & x \in E^c \end{cases}.$$

Indeed $\nu(A) = \nu(A \cap E) + \nu(A \cap E^c)$. On $A \cap E$ we have $\nu(A \cap E) = \int_{A \cap E} g \, \mathrm{d}\mu = \int_{A \cap E} f \, \mathrm{d}\mu$. On $A \cap E^c$ either $\mu(A \cap E^c) = 0$ in which case $\int_{A \cap E^c} f \, \mathrm{d}\mu = 0$ and $\nu(A \cap E^c) = 0$ by $\nu \ll \mu$, or $\mu(A \cap E^c) > 0$ in which case $\nu(A \cap E^c) = \infty$ and $\int_{A \cap E^c} f \, \mathrm{d}\mu = \infty$.

Folland 3.15. A measure μ on $(X, \mathcal{M})$ is called decomposable if there is $\mathcal{F} \subset \mathcal{M}$ with: (i) $\mu(F) < \infty$ for all $F \in \mathcal{F}$; (ii) the members of $\mathcal{F}$ are disjoint and their union is X; (iii) if $\mu(E) < \infty$, then $\mu(E) = \sum_{F \in \mathcal{F}} \mu(E \cap f)$; (iv) if $E \subset X$ and $E \cap F \in \mathcal{M}$ for all $F \in \mathcal{F}$, then $E \in \mathcal{M}$.

(a) If μ is σ-finite, then by definition there is $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ such that $\bigcup_{n=1}^\infty F_n = X$ is a disjoint union and $\mu(F) < \infty$ satisfying (i) and (ii). Moreover, (iii) and (iv) follow from the other two properties. If $E \subset X$ and $E \cap F_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, then $M \ni \bigcup_{n=1}^\infty (E \cap F_n) = E \cap \bigcup_{n=1}^\infty F_n = E \cap X = E$. And $\mu(E) = \mu\left(\bigcup_{n=1}^\infty E \cap F_n\right) = \sum_{n=1}^\infty \mu(E \cap F_n)$.

(b) Let μ be decomposable, and let $\mathcal{F} \subset \mathcal{M}$ be a decomposition satisfying (i)-(iv). For each $F \in \mathcal{F}$ define the restricted measures $\mu_F(A) = \mu(A \cap F)$ and $\nu_F(A) = \nu(A \cap F)$. By (i) $\mu_F(F) < \infty$ and clearly $\nu_F \ll \mu_F$. Hence 3.14 yields an extended μ_F-integrable function $f_F : F \to [-\infty, \infty]$ such that $\nu_F(A) = \int_A f_F \, \mathrm{d}\mu_F$. Define the function $f : X \to [-\infty, \infty]$ by

$$f(x) = \sum_{f \in \mathcal{F}} f_F(x) \chi_F(x).$$

Now let $E \in \mathcal{M}$ be σ-finite for μ. Write $E = \bigcup_{n=1}^\infty E_n$ with $\mu(E_n)$ and assume w.l.o.g. that E_n are disjoint. By (iii)

$$\mu(E_n) = \sum_{F \in \mathcal{F}} \mu(E_n \cap F),$$

so for each n only countably many F satisfy $\mu(E_n \cap F) > 0$. Therefore the collection $\mathcal{F}_E = \{F \in \mathcal{F} : \mu(E \cap F) > 0\}$ is countable (as a countable union of countable sets). Consequently:

$$\int_E f \, \mathrm{d}\mu = \int_E \sum_{F \in \mathcal{F}_E} f_F \chi_F \, \mathrm{d}\mu = \sum_{F \in \mathcal{F}_E} \int_E f_F \chi_F \, \mathrm{d}\mu = \sum_{F \in \mathcal{F}_E} \int_E f_F \, \mathrm{d}\mu_F = \sum_{F \in \mathcal{F}_E} \nu(E \cap F) = \nu(E).$$

Note we can ignore terms where $\mu(E \cap F) = 0$ since $\nu \ll \mu$ and so $\nu(E \cap F) = 0$ anyways.

Folland 3.16. We first show that $\mathrm{d}\nu / \mathrm{d}\mu = f/(1-f)$. Indeed for $A \in \mathcal{M}$:

$$\int_A (1 - f) \, \mathrm{d}\lambda + \nu(A) = \int_A 1 \, \mathrm{d}\lambda - \nu(A) + \nu(A) = \lambda(A) = \mu(A) + \nu(A).$$

So in particular $\mathrm{d}\mu / \mathrm{d}\lambda = (1 - f)$. And since $\mu \ll \lambda$ and $\lambda \ll \mu$ we have $\mathrm{d}\lambda / \mathrm{d}\mu = 1/(1 - f)$. So finally

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = \frac{f}{1 - f}.$$

Now we show that $0 \le f < 1$ μ-a.e. Since ν and λ are positive measures, we must have $f \ge 0$ λ-a.e., that is $\lambda(\{x \in X : f(x) < 0\}) = 0$. And since $\mu \ll \lambda$ we have that $f \ge 0$ μ-a.e. also. Now suppose that $f \ge 1$ on some E with $\mu(E) > 0$. Then

$$\nu(E) = \int_E f \, \mathrm{d}\lambda = \mu(E) + \nu(E).$$

But this implies that $\mu(E) \le 0$, a contradiction.

Folland 3.17. Define the measure $\lambda(E) = \int_E f \, \mathrm{d}\mu$ which is finite since $f \in L^1(\mu)$. Define $\rho = \lambda|_{\mathcal{N}}$. Clearly $\lambda \ll \mu$, and since $\nu = \mu|_{\mathcal{N}}$ we have that $\rho \ll \nu$. So by Radon-Nikodym there is an extended ν-integrable function g such that $\rho(E) = \int_E g \, \mathrm{d}\nu$ for all $E \in \mathcal{N}$. Moreover $g \in L^1(\nu)$ since $\rho(X) < \infty$ and g is unique up to equality ν-a.e. by Radon-Nikodym. Since ρ is the restriction of λ we have $\int_E f \, \mathrm{d}\mu = \int_E f \, \mathrm{d}\nu$ for all $E \in \mathcal{N}$.

Folland 3.18. Let ν be a complex measure and suppose $\mathrm{d}\nu = f \, \mathrm{d}\mu$. Recall the definitions

$$L^1(\nu) = L^1(\nu_r) \cap L^1(\nu_i) = \left\{ \text{measurable } f : \int |f| \, \mathrm{d}\nu_{r,i}^{\pm} < \infty \right\}$$

$$L^1(|\nu|) = \left\{ \text{measurable } f : \int |f| \, \mathrm{d}|\nu| < \infty \right\}, \qquad \text{where } \mathrm{d}|\nu| = |f| \, \mathrm{d}\mu.$$

We first show that $L^1(|\nu|) \subset L^1(\nu)$. If $g \in L^1(|\nu|)$, then

$$\infty > \int |g| \, \mathrm{d}|\nu| = \int |g||f| \, \mathrm{d}\mu \geq \left| \int |g| f \, \mathrm{d}\mu \right| \geq \int |g| \, \mathrm{d}\nu.$$

Now we show that $L^1(\nu) \subset L^1(|\nu|)$. If $g \in L^1(\nu)$, then

$$\int |g| \, \mathrm{d}\nu_r^{\pm} < \infty \qquad \text{and} \qquad \int |g| \, \mathrm{d}\nu_i^{\pm} < \infty.$$

Moreover we have that $|f| \leq |\mathrm{Re} f| + |\mathrm{Im} f| = (\mathrm{Re} f)^+ + (\mathrm{Re} f)^- + (\mathrm{Im} f)^+ + (\mathrm{Im} f)^-$. Hence

$$\int |f||g| \, \mathrm{d}\mu \leq \int |g|(\mathrm{Re} f)^+ \, \mathrm{d}\mu + \int |g|(\mathrm{Re} f)^- \, \mathrm{d}\mu + \int |g|(\mathrm{Im} f)^+ \, \mathrm{d}\mu + \int |g|(\mathrm{Im} f)^- \, \mathrm{d}\mu.$$

These terms are all $\int |g| \, \mathrm{d}\nu_{i,r}^{\pm}$ which are finite so:

$$\int |g| \, \mathrm{d}|\nu| = \int |g||f| \, \mathrm{d}\mu < \infty.$$

Finally we write

$$\left| \int g \, \mathrm{d}\nu \right| = \left| \int g f \, \mathrm{d}\mu \right| \leq \int |g||f| \, \mathrm{d}\mu = \int |g| \, \mathrm{d}|\nu|.$$