Real Analysis

MATH 507

Winter 2025

Introduction

This set of notes is transcribed from UBC's MATH 507 Measure Theory course. If any errors are found, please feel free to email me at `nathan.cantafio@stat.ubc.ca`

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1.1 σ-algebras

From now on, X is a non-empty set and denote the power set of X by $\mathcal{P}(X)$.

Definition 1.1. A non-empty $\mathcal{A} \subset \mathcal{P}(X)$ is an **algebra** if

(i) $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$

(ii) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$

Note that these conditions imply (1) $\emptyset = A \cap A^c \in \mathcal{A}$, (2) $X = A \cup A^c \in \mathcal{A}$, and (3) finite unions and intersections of $A_i \in \mathcal{A}$ belong to $\mathcal{A}$.

Definition 1.2. A **σ-algebra** is an algebra such that $\{A_i\}_{i\in\mathbb{N}} \subset \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Some examples of σ-algebras include $\{\emptyset, X\}$, $\mathcal{P}(X)$, and $\mathcal{A} = \{E \subset X : E \text{ is countable or } E^c \text{ is countable}\}$.

Observation 1.3. The arbitrary intersection of σ-algebras is a σ-algebra.

Proof. Let $\mathcal{I}$ be any index set and let $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ be a collection of σ-algebras. Define $\mathcal{A} = \bigcap_{i\in\mathcal{I}} \mathcal{A}_i$. Since $\emptyset \in \mathcal{A}_i$ for all $i \in \mathcal{I}$ so is $\emptyset \in \mathcal{A}$, so $\mathcal{A}$ is non-empty. Now let $\{E_n\}_{n\in\mathbb{N}}$ be in $\mathcal{A}$. Then $E_n \in \mathcal{A}_i$ for all $i \in \mathcal{I}$. Hence $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}_i$ for all $i \in \mathcal{I}$, and so $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}$. $\square$

Definition 1.4. Let $\mathcal{E} \in \mathcal{P}(X)$. The σ-algebra $\mathcal{M}(\mathcal{E})$ **generated by** $\mathcal{E}$ is the smallest σ-algebra containing $\mathcal{E}$. Namely:

$$\mathcal{M}(\mathcal{E}) = \bigcap_{\substack{\mathcal{E}\subset\mathcal{A} \\ \mathcal{A} \text{ is a } \sigma\text{-algebra}}} \mathcal{A}.$$

Definition 1.5. Let X be a topological space. The **Borel σ-algebra** $\mathcal{B}(X)$ on X is the σ-algebra generated by the open sets of X. Note that $\mathcal{B}(X)$ contains all open sets, all closed sets, all countable intersections of open sets (so-called G_δ sets), all countable unions of closed sets (so-called F_σ sets), and so on...

Lemma 1.6. Let $\mathcal{E}, \mathcal{F} \subset \mathcal{P}(X)$. Then

(i) $\mathcal{E} \subset \mathcal{M}(\mathcal{F}) \implies \mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$

(ii) $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$ and $\mathcal{F} \subset \mathcal{M}(\mathcal{E})$ together imply $\mathcal{M}(\mathcal{E}) = \mathcal{M}(\mathcal{F})$

Proof. Notice that (i) immediately implies (ii) and that (i) follows from minimality of $\mathcal{M}(\mathcal{E})$. $\square$

Observation 1.7. $\mathcal{B}(\mathbb{R})$ is generated by any of the following families:

(i) $\{(a, b) : a < b\}$

(ii) $\{[a, b] : a < b\}$

(iii) $\{[a, b) : a < b\}$ and $\{(a, b] : a < b\}$

(iv) $\{(-\infty, b) : b \in \mathbb{R}\}$ and $\{(a, \infty) : a \in \mathbb{R}\}$

(v) $\{(-\infty, b] : b \in \mathbb{R}\}$ and $\{[a, \infty) : a \in \mathbb{R}\}$

We will only prove (i).

Proof. Let $\mathcal{T}$ be the collection of open sets and $\mathcal{E} = \{(a,b) : a < b\}$. Notice $\mathcal{E} \subset \mathcal{T} \subset \mathcal{M}(\mathcal{T}) = \mathcal{B}(\mathbb{R})$ so $\mathcal{M}(\mathcal{E}) \subset \mathcal{B}(\mathbb{R})$ by lemma 1.1. To show the reverse inclusion it suffices to show that $\mathcal{T} \subset \mathcal{M}(\mathcal{E})$. Namely that any open set can be written as the countable union of open intervals. Let $A \in \mathcal{T}$. Let $x \in A$. Since A is open there exists $a < b$ such that $x \in (a,b) \subset A$. There then exists $p, q \in \mathbb{Q}$ such that $a < p < x < q < b$ and hence $A \subset \bigcup_{\substack{p,q\in\mathbb{Q} \\ (p,q)\subset A}} (p,q)$ which is countable. Hence $A \in \mathcal{M}(\mathcal{E})$. $\square$

1.2 Measures

Definition 1.8. A **measure** μ on a σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$ is a function $\mu : \mathcal{M} \to [0,\infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) If $\{E_i\}_{i\in\mathbb{N}}$ is a countable collection of disjoint sets, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Note that on an algebra one can only define a finitely additive measure.

Definition 1.9. A measure μ on $\mathcal{M}$ is called:

(i) **finite** if $\mu(X) < \infty$

(ii) **σ-finite** if there is $\{E_i\}_{i\in\mathbb{N}}$ in $\mathcal{M}$ such that $X = \bigcup_{i=1}^{\infty} E_i$ and $\mu(E_i) < \infty$

(iii) **semi-finite** if for each $E \in \mathcal{M}$ such that $\mu(E) = \infty$, there is $F \in \mathcal{M}$ such that $0 < \mu(F) < \infty$ and $F \subset E$

(iv) **Borel** if X is a topological space and $\mathcal{M} = \mathcal{B}(X)$

Example 1.10.

(i) $\mathcal{M} = \mathcal{P}(X)$ and $\mu(E) = \#$ of points in E is called the **counting measure** on X

(ii) For any σ-algebra $\mathcal{M} \subset \mathcal{P}(X)$, for $x \in X$:

$$\mu_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

(iii) For $\mathcal{M} = \{E \in \mathcal{P}(X) : E \text{ is countable or } E \text{ is co-countable}\}$, then

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{if } E \text{ is co-countable} \end{cases}$$

is a measure. Indeed:

- If $\{E_i\}_{i\in\mathbb{N}}$ are all countable then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = 0$ and $\mu(E_i) = 0$ for all $i \in \mathbb{N}$.
- If E_{i_0} is co-countable and $\{E_i\}_{i\in\mathbb{N}\setminus\{i_0\}}$ are all countable then $\sum_{i=1}^{\infty} \mu(E_i) = \mu(E_{i_0}) = 1$ while $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = 1$ since the union is co-countable.
- There cannot be two disjoint co-countable sets E, F since $F \subset E^c$.

Theorem 1.11. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $E, F \in \mathcal{M}$ and let $\{E_i\}_{i\in\mathbb{N}} \subset \mathcal{M}$.

(i) *(Monotonicity)*: $E \subset F \implies \mu(E) \leq \mu(F)$

(ii) *(Subadditivity)*: $\mu\left(\bigcup_{i=1}^\infty E_i\right) \leq \sum_{i=1}^\infty \mu(E_i)$

(iii) *(Continuity from below)*: If $E_1 \subset E_2 \subset \cdots$ then $\mu\left(\bigcup_{i=1}^\infty E_i\right) = \lim_{i\to\infty} \mu(E_i)$

(iv) *(Continuity from above)*: If $\mu(E_1) < \infty$ and $E_1 \supset E_2 \supset \cdots$ then $\mu\left(\bigcap_{i=1}^\infty E_i\right) = \lim_{i\to\infty} \mu(E_i)$

Note that assumption in (iv) that $\mu(E_1) < \infty$ is necessary. Consider the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ equipped with the counting measure. Let $E_i = \{n \in \mathbb{N} : n \geq i\}$. Then $\mu(E_i) = \infty$ for each i, however $\bigcap_{i=1}^\infty E_i = \emptyset$ and so $\mu\left(\cap_{i=1}^\infty E_i\right) = 0 \neq \infty = \lim_{i\to\infty} E_i$.

Proof.

(i) $\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$.

(ii) Let $F_i = E_i \setminus \left(\bigcup_{j=1}^{i-1} E_j\right)$, then $\bigcup_{i=1}^\infty E_i = \bigcup_{i=1}^\infty F_i$ and so

$$\mu\left(\bigcup_{i=1}^\infty E_i\right) = \mu\left(\bigcup_{i=1}^\infty F_i\right) = \sum_{i=1}^\infty \mu(F_i) \leq \sum_{i=1}^\infty \mu(E_i)$$

where the last inequality follows from (i) since $F_i \subset E_i$.

(iii) Writing $\bigcup_{i=1}^\infty E_i = E_1 \cup \left[\bigcup_{j=2}^\infty E_j \setminus E_{j-1}\right]$ is a union of disjoint sets and so

$$\mu\left(\bigcup_{i=1}^\infty E_i\right) = \mu(E_1) + \sum_{j=2}^\infty \mu(E_j \setminus E_{j-1}) = \lim_{n\to\infty}\left[\mu(E_1) + \sum_{j=2}^n \mu(E_j \setminus E_{j-1})\right] = \lim_{n\to\infty} \mu(E_n).$$

(iv) Let $F_i = E_1 \setminus E_i$. Then $F_1 \subset F_2 \subset \cdots$. So by (iii) we have $\mu\left(\bigcup_{i=1}^\infty F_i\right) = \lim_{i\to\infty} \mu(F_i)$. Since $\mu(E_1) = \mu(E_i) + \mu(F_i)$ we can subtract to obtain $\mu(F_i) = \mu(E_1) - \mu(E_i)$. Therefore

$$\mu\left(\bigcup_{i=1}^\infty F_i\right) = \lim_{i\to\infty} \mu(F_i) = \mu(E_1) - \lim_{i\to\infty} \mu(E_i).$$

On the other hand:

$$\bigcup_{i=1}^\infty F_i = \bigcup_{i=1}^\infty E_1 \cap E_i^c = E_1 \cap \left(\bigcup_{i=1}^\infty E_i^c\right) = E_1 \cap \left(\bigcap_{i=1}^\infty E_i\right)^c = E_1 \setminus \bigcap_{i=1}^\infty E_i,$$

hence

$$\mu\left(\bigcup_{i=1}^\infty F_i\right) = \mu(E_1) - \mu\left(\bigcap_{i=1}^\infty E_i\right).$$

Equating both expressions yields the desired result.

□

Definition 1.12. Let $(X, \mathcal{M}, \mu)$ be a measure space

(i) A **null set** is a set $E \subset \mathcal{M}$ such that $\mu(E) = 0$

(ii) If $f : X \to \{\text{true}, \text{false}\}$ is a statement about points in X and $\mu(\{x \in X : f(x) = \text{false}\}) = 0$ then f is said to be **true almost everywhere**, usually written "true a.e."

Definition 1.13. A measure space $(X, \mathcal{M}, \mu)$ is **complete** if for all $N \in \mathcal{M}$, $\mu(N) = 0$ we have $Z \subset N \implies Z \in \mathcal{M}$. In other words, if all subsets of null sets are measurable.

Theorem 1.14. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$. And define $\bar{\mathcal{M}} = \{E \cup Z : E \in \mathcal{M}, Z \subset N, \text{ and } N \in \mathcal{N}\}$. Finally extend μ to $\bar{\mathcal{M}}$ as $\bar{\mu} : \bar{\mathcal{M}} \to [0, \infty]$ such that $\bar{\mu}(E \cup Z) = \mu(E)$. Then:

(i) $\bar{\mathcal{M}}$ is a σ-algebra

(ii) $\bar{\mu}$ is a complete measure on $\bar{\mathcal{M}}$ called the **completion of** μ

(iii) $\bar{\mu}$ is the unique extension of μ to $\bar{\mathcal{M}}$

Proof.

(i) We need to show that $\bar{\mathcal{M}}$ is non-empty, closed under countable union, and closed under complement. It is non-empty since $\mathcal{M}$ is non-empty. Now let $\{E_i\}_{i \in \mathbb{N}} \subset \mathcal{M}$, $\{N_i\}_{i \in \mathbb{N}} \subset \mathcal{N}$ and $Z_i \subset N_i$. Then

$$\bigcup_{i=1}^{\infty} (E_i \cup Z_i) = \left(\bigcup_{i=1}^{\infty} E_i \right) \cup \left(\bigcup_{i=1}^{\infty} Z_i \right).$$

By subadditivity we have $\mu\left(\bigcup_{i=1}^{\infty} N_i\right) \leq \sum_{i=1}^{\infty} \mu(N_i) = 0$, hence the $\bigcup_{i=1}^{\infty} N_i$ is a null-set. Then since $\bigcup_{i=1}^{\infty} Z_i \subset \bigcup_{i=1}^{\infty} N_i$ and $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$ we get that $\bar{\mathcal{M}}$ is closed under countable union. Next let $E \in \mathcal{M}$, $N \in \mathcal{N}$ and $Z \subset N$. Let $N' = N \setminus E = N \cap E^c \in \mathcal{M}$ and let $Z' = Z \setminus E \subset N'$. Now $X = E \cup Z' \cup (N' \setminus Z') \cup (E \cup N')^c$ is a disjoint union so in particular $(E \cup Z')^c = (E \cup N')^c \cup (N' \setminus Z')$. And since $\mu(N') = 0$ by monotonicity, we have $N' \in \mathcal{N}$. In particular $N' \setminus Z \subset N'$ and $(E \cup N')^c \in \mathcal{M}$. So we conclude $(E \cup Z')^c \in \bar{\mathcal{M}}$.

(ii) $\bar{\mu}$ is well-defined. That is, if $E \cup Z = E' \cup Z'$ then $\mu(E) = \bar{\mu}(E \cup Z) = \bar{\mu}(E' \cup Z') = \mu(E')$. Indeed: $\mu(E) = \mu(E \cap E') + \mu(E \setminus E')$. Now $E \setminus E' \subset Z' \subset N'$ so by monotonicity $\mu(E \setminus E') = 0$. Namely by symmetry: $\mu(E) = \mu(E \cap E') = \mu(E')$.

Let $\bar{N} \in \mathcal{M}$ with $\bar{\mu}(\bar{N}) = 0$. Write $\bar{N} = E \cup \bar{Z}$ with $E \in \mathcal{M}$ and $\bar{Z} \subset N_0 \in \mathcal{N}$. In fact, $\mu(E) = \bar{\mu}(\bar{N}) = 0$. Notice that $E \cup N_0 \in \mathcal{M}$ and that by subadditivity:

$$\mu(E \cup N_0) \leq \mu(E) + \mu(N_0) = 0.$$

In particular, $E \cup N_0$ is a measurable null set containing $\bar{N}$. Now take any $Z \subset \bar{N}$. Since $\emptyset \in \mathcal{M}$ and $Z \subset E \cup N_0$ also, we can write $Z = \emptyset \cup Z \in \bar{\mathcal{M}}$.

(iii) Let μ' be another extension of μ to $\bar{\mathcal{M}}$. If $E \in \mathcal{M}$ then $\mu'(E) = \mu(E) = \bar{\mu}(E)$. Otherwise if $\bar{E} = E \cup Z$, then $\mu(E) = \mu'(E) \leq \mu'(\bar{E}) \leq \mu'(E \cup N) \leq \mu'(E) + \mu'(N) = \mu(E) + \mu(N) = \mu(E)$. Thus $\mu'(\bar{E}) = \mu(E) = \bar{\mu}(\bar{E})$.

□

1.3 Outer measures

Definition 1.15. An **outer measure** on X is a function $\mu^\star : \mathcal{P}(X) \to [0, \infty]$ such that

(i) $\mu^\star(\emptyset) = 0$

(ii) If $A \subset B$, then $\mu^\star(A) \leq \mu^\star(B)$

(iii) For any countable collection $\{A_i\}_{i \in \mathbb{N}}$, $A_i \subset X$ we have $\mu^\star\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^\star(A_i)$.

Note that $\mu^\star$ is defined on all subsets of X but only satisfies monotonicity & countable subadditivity.

Proposition 1.16. Let $S \subset \mathcal{P}(X)$ and $\rho : S \to [0,\infty]$ be such that $\emptyset \in S$, $X \in S$ and $\rho(\emptyset) = 0$. For any $A \subset X$, let

$$\mu^\star(A) = \inf\left\{\sum_{i=1}^{\infty} \rho(S_i) : \{S_i\}_{i\in\mathbb{N}} \subset S \text{ and } A \subset \bigcup_{i=1}^{\infty} S_i\right\}.$$

Then $\mu^\star$ is an outer measure.

Proof.

(i) $\mu^\star(\emptyset) = 0$ since $\emptyset \in S$ and $\rho(\emptyset) = 0$.

(ii) Let $A \subset B$. If $\bigcup_{i=1}^{\infty} S_i \supset B$, then it also covers A. That is the set of all covers of B is a subset of the set of all covers of A. So taking the infinum over all such covers yields $\mu^\star(A) \le \mu^\star(B)$.

(iii) Let $\varepsilon > 0$. Let $\{A^{(n)}\}_{n\in\mathbb{N}}$ be such that $\mu^\star(A^{(n)}) < \infty$ [if $\mu^\star(A^{(n_0)}) = \infty$ the claim is trivial]. Let $\{S_i^{(n)}\}_{i\in\mathbb{N}}$ cover $A^{(n)}$ be such that $\sum_{i=1}^{\infty} \rho\left(S_i^{(n)}\right) \le \mu^\star\left(A^{(n)}\right)) + \varepsilon/2^n$ [such a cover must exist by definition of infinum]. Now $\{S_i^{(n)}\}_{(i,n)\in\mathbb{N}\times\mathbb{N}} \subset S$ and $\bigcup_{n=1}^{\infty} A^{(n)} \subset \bigcup_{i,n\in\mathbb{N}\times\mathbb{N}} S_i^{(n)}$ so then:

$$\mu^\star\left(\bigcup_{n=1}^{\infty} A^{(n)}\right) \le \sum_{i,n=1}^{\infty} \rho\left(S_i^{(n)}\right) \le \sum_{n=1}^{\infty} \mu^\star\left(A^{(n)}\right) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we conclude that $\mu^\star\left(\bigcup_{n=1}^{\infty} A^{(n)}\right) \le \sum_{n=1}^{\infty} \mu^\star\left(A^{(n)}\right)$.

□

Example 1.17. The **Lebesgue outer measure** on $\mathbb{R}$: let $S = \{(a,b) : -\infty \le a \le b \le \infty\}$, $\rho(\emptyset) = 0$, $\rho(\mathbb{R}) = \infty$ and $\rho((a,b)) = b - a$.

Definition 1.18. Let $\mu^\star$ be an outer measure on X, then $A \subset X$ is **$\mu^\star$-measurable** if

$$\mu^\star(E) = \mu^\star(E \cap A) + \mu^\star(E \cap A^c)$$

for all $E \subset X$.

Note 1.19.

(i) $\mu^\star(E) \le \mu^\star(E \cap A) + \mu^\star(E \cap A^c)$ by subadditivity

(ii) If $\mu^\star(E) = \infty$, then "$\ge$" is trivial. Therefore to verify that A is $\mu^\star$-measurable, it suffices to verify that $\mu^\star(E) \ge \mu^\star(E \cap A) + \mu^\star(E \cap A^c)$ for all $E \subset X$ with $\mu^\star(E) < \infty$.

(iii) Some motivation for the definition: if $A \subset E$ then

$$\begin{aligned}\text{"inner volume of } A\text{"} &= \sup\{\text{vol}(S) : S \text{ is simple and } S \subset A\} \\ &= \sup\{\text{vol}(E \setminus T) : T \text{ simple and } E \setminus A \subset T\} \\ &= \text{vol}(E) - \inf\{\text{vol}(T) : T \text{ simple and } E \setminus A \subset T\} \\ &= \mu^\star(E) - \mu^\star(E \setminus A)\end{aligned}$$

The volume of A is well-defined if its "inner volume" equals its "outer volume". That is if $\mu^\star(E) - \mu^\star(E \setminus A) = \mu^\star(A)$. Or in other words: $\mu^\star(E) = \mu^\star(E \cap A) + \mu^\star(E \cap A^c)$.

Theorem 1.20. *(Carathéodory)* Let $\mu^\star$ be an outer measure and let $\mathcal{M}^\star = \{A \subset X : A \text{ is } \mu^\star\text{-measureable}\}$. Then

(i) $\mathcal{M}^\star$ is a σ-algebra

(ii) The restriction $\mu^\star \restriction \mathcal{M}^\star$ is a complete measure

(iii) If $N \subset X$ is such that $\mu^\star(N) = 0$, then $N \in \mathcal{M}^\star$

Proof.

(i)
- $\mathcal{M}^\star$ is non-empty since $\emptyset \in \mathcal{M}^\star$. Indeed $\mu^\star(E \cap \emptyset) + \mu^\star(E \cap \emptyset^c) = \mu^\star(E)$ for all $E \subset X$.
- $\mathcal{M}^\star$ is closed under complement since $(A^c)^c = A$.
- $\mathcal{M}^\star$ is closed under finite unions. It suffices to show that for $A, B \in \mathcal{M}^\star$ and $E \subset X$ with $\mu(E) < \infty$ that $\mu^\star(E) \geq \mu^\star(E \cap (A \cup B)) + \mu^\star(E \cap (A \cup B)^c)$. Indeed using first that $A \in \mathcal{M}^\star$ and then $B \in \mathcal{M}^\star$ we get

  $$\mu^\star(E) \geq \mu^\star(E\cap A)+\mu^\star(E\cap A^c) \geq \mu^\star(E\cap A\cap B)+\mu^\star(E\cap A\cap B^c)+\mu^\star(E\cap A^c\cap B)+\mu^\star(E\cap A^c\cap B^c).$$

  Then since $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, we get by subadditivity that

  $$\mu^\star(E) \geq \mu^\star(E \cap (A \cup B)) + \mu^\star(E \cap (A \cup B)^c).$$

- $\mu^\star$ is finitely additive. Let $E \subset X$ and let $A, B \in \mathcal{M}^\star$ be disjoint. Then since $A \in \mathcal{M}^\star$:

  $$\begin{aligned} \mu^\star(E \cap (A \cup B)) &= \mu^\star(E \cap (A \cup B) \cap A) + \mu^\star(E \cap (A \cup B) \cap A^c) \\ &= \mu^\star(E \cap A) + \mu^\star(E \cap B). \end{aligned} \tag{$\star$}$$

  Letting $E = X$ yields $\mu^\star(A \cup B) = \mu^\star(A) + \mu^\star(B)$.
- $\mathcal{M}^\star$ is σ-algebra. It suffices to show that $\mathcal{M}^\star$ is closed under countable disjoint unions. Let $\{A_i\}_{i\in\mathbb{N}} \subset \mathcal{M}^\star$ be disjoint. Let $E \subset X$. For each $n \in \mathbb{N}$, $\bigcup_{i=1}^n A_i \in \mathcal{M}^\star$ by the above. Hence

  $$\begin{aligned} \mu^\star(E) &= \mu^\star\left(E \cap \left(\bigcup_{i=1}^n A_i\right)\right) + \mu^\star\left(E \cap \left(\bigcup_{i=1}^n A_i\right)^c\right) \\ &\overset{(\star)}{=} \sum_{i=1}^n \mu^\star(E \cap A_i) + \mu^\star\left(E \cap \left(\bigcup_{j=1}^n A_j\right)^c\right) \\ &\geq \sum_{i=1}^n \mu^\star(E \cap A_i) + \mu^\star\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right)^c\right) \end{aligned}$$

  where the inequality follows from monotonicity. Then let $n \to \infty$ and use subadditivity to obtain:

  $$\mu^\star(E) \geq \mu^\star\left(E \cap \left(\bigcup_{i=1}^\infty A_i\right)\right) + \mu^\star\left(E \cap \left(\bigcup_{i=1}^\infty A_i\right)^c\right) \geq \mu^\star(E).$$

  Hence all inequalities are equalities and $\bigcup_{i=1}^\infty A_i \in \mathcal{M}^\star$.

(ii) • $\mu^\star \restriction \mathcal{M}^\star$ is countably additive. Applying $(\star)$ n times we get:

$$\mu^\star\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \sum_{i=1}^{n} \mu^\star(E \cap A_i) + \mu^\star\left(E \cap \left(\bigcup_{i=n+1}^{\infty} A_i\right)\right) \geq \sum_{i=1}^{n} \mu^\star(E \cap A_i).$$

Taking $n \to \infty$ yields countable superadditivity. Since $\mu^\star$ is countably subadditive also, it is countably additive.

• $\mu^\star \restriction \mathcal{M}^\star$ is complete. Let $N \in \mathcal{M}^\star$ be such that $\mu^\star(N) = 0$ and let $Z \subset N$. Let $E \subset X$. Then using subadditivity and then monotonicity:

$$\begin{aligned} \mu^\star(E) &\leq \mu^\star(E \cap Z) + \mu^\star(E \cap Z^c) \\ &\leq \mu^\star(N) + \mu^\star(E) \\ &= \mu^\star(E) \end{aligned}$$

So all inequalities are equalities and Z is $\mu^\star$-measurable. Namely $Z \in \mathcal{M}^\star$.

(iii) We did not need $N \in \mathcal{M}^\star$ for the above. Taking $Z = N$ shows that $\mu^\star(N) = 0 \implies N \in \mathcal{M}^\star$.

□

This is a powerful result, that lets us construct a measure from an outer measure. However we know little about that measure or even the σ-algebra on which it is defined. We want to be able to say more about the measure and associated σ-algebra we construct. So let's start from another angle.

Definition 1.21. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra. A **premeasure** on $\mathcal{A}$ is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ such that:

(i) $\mu_0(\phi) = 0$

(ii) If $\{A_i\}_{i \in \mathbb{N}}$ is a countable collection of disjoint sets in $\mathcal{A}$ and if $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, then

$$\mu_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_0(A_i)$$

Note that this definition arrises because we want to easily be able to construct premeasures and to make it possible to enlarge $\mathcal{A}$ to a σ-algebra $\mathcal{M}$ and to extend μ_0 to a measure μ on $\mathcal{M}$.

Proposition 1.22. Let $F : \mathbb{R} \to \mathbb{R}$ be nondecreasing and right continuous. Let $F(\pm\infty) = \lim_{x \to \pm\infty} F(x)$. Let $\mathcal{A} = \{\emptyset\} \cup \{\bigcup_{j=1}^{n} (a_j, b_j] : n \in \mathcal{N}, -\infty \leq a_1 < b_1 < a_2 < \cdots < b_n \leq \infty\}$. And let $\mu_0(\emptyset) = 0$, $\mu_0\left(\bigcup_{j=1}^{n} (a_j, b_j]\right) = \sum_{j=1}^{n} (F(b_j) - F(a_j))$. Then $\mathcal{A}$ is an algebra and μ_0 is a premeasure. Note that when $b = \infty$ we mean (a, ∞) by $(a, b]$.

Proof. Clearly $\mathcal{A}$ is closed under finite union. It is also closed under complement since we have both $\emptyset^c = (-\infty, \infty] = \mathbb{R}$ and $(\bigcup_{i=1}^{n} (a_i, b_i])^c = (-\infty, a_1] \cup (b_1, a_2] \cup \cdots \cup (b_n, \infty] \in \mathcal{A}$. Now to show that μ_0 is a premeasure.

(i) $\mu_0(\emptyset) = 0$ by definition.

(ii) Let $\{\mathcal{I}_i\}_{i\in\mathbb{N}}$ be a countable family of disjoint sets in $\mathcal{A}$ such that $\mathcal{I} = \bigcup_{i=1}^{\infty} \mathcal{I}_i \in \mathcal{A}$. Without loss of generality supposes that $\mathcal{I}_i \neq \emptyset$, $\mathcal{I}_i = (a_i, b_i]$ and that $\mathcal{I} = (a, b]$. We need to show that $\sum_{i=1}^{\infty} F(b_i) - F(a_i) = F(b) - F(a)$. Let $n \in \mathbb{N}$. Then by relabeling we have $a \le a_1 < b_1 \le a_2 < b_2 \cdots < b_n \le b$ and

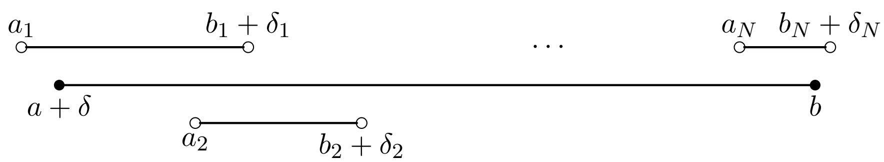
$$\begin{aligned}\sum_{i=1}^{n} \mu_0(\mathcal{I}_i) &= \underbrace{F(b_n)}_{\le F(B)} \underbrace{-F(a_n) + F(b_{n-1})}_{\le 0} \cdots \underbrace{-F(a_1)}_{\le F(a)} \\ &\le F(b) - F(a) \\ &= \mu_0(\mathcal{I})\end{aligned}$$

Since this holds for any $n \in \mathbb{N}$, it also holds in the limit. Now we show inequality in the other direction. First suppose a and b are finite. Let $\varepsilon > 0$. By right continuity of F we have

$$\text{there exists } \delta > 0 \text{ such that } F(a+\delta) - F(a) < \varepsilon \tag{$\star$}$$

$$\text{there exists } \delta_m > 0 \text{ such that } F(b_m + \delta_m) - F(b_m) < \varepsilon/2^m \tag{$\star\star$}$$

Now $\{(a_m, b_m + \delta_m)\}_{m\in\mathbb{N}}$ covers $[a+\delta, b]$. By compactness we can choose a finite sub-cover:



Then using the fact that F is non-decreasing

$$\begin{aligned}\mu_0(\mathcal{I}) &= F(b) - F(a) \\ &\overset{(\star)}{<} F(b) - F(a+\delta) + \varepsilon \\ &\le F(b_N + \delta_N) - F(a_N) + \varepsilon \\ &= \underbrace{F(b_N + \delta_n)}_{\le F(b_N) + \varepsilon/2^N} - F(a_N) + \sum_{i=1}^{N-1} \Big[\underbrace{F(a_{i+1})}_{\substack{\le F(b_i + \delta_i) \\ \overset{(\star)}{<} F(b_i) + \varepsilon/2^i}} - F(a_i)\Big] + \varepsilon \\ &< \sum_{i=1}^{N} F(b_i) - F(a_i) + 2\varepsilon\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, and $\sum_{i=1}^{N} \le \sum_{i=1}^{\infty}$ we obtain $\sum_{i=1}^{\infty} \mu_0(\mathcal{I}_i) \ge F(b) - F(a)$. Now if either $a = -\infty$ or $b = \infty$. Let $M > 0$. Then by the above:

$$F(\min\{b, M\}) - F(\max\{-M, a\}) \le \sum_{i=1}^{\infty} \mu_0(\mathcal{I}_i),$$

then let $M \to \infty$.

$\square$

Theorem 1.23. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, and let $\mathcal{M} = \mathcal{M}(\mathcal{A})$ be the σ-algebra generated by $\mathcal{A}$. Let μ_0 be a premeasure on $\mathcal{A}$. Let $\mu^\star$ be the outer measure induced by μ_0, and $\mathcal{M}^\star$ the set of $\mu^\star$-measurable sets. Then:

(i) $\mu^\star \restriction \mathcal{A} = \mu_0$

(ii) $\mathcal{M} \subset \mathcal{M}^\star$ and $\mu := \mu^\star \restriction \mathcal{M}$ extends μ_0. [All sets in $\mathcal{M}$ are $\mu^\star$-measurable and $\mu \restriction \mathcal{A} = \mu_0$]

(iii) If ν is any other measure on $\mathcal{M}$ such that $\nu \restriction \mathcal{A} = \mu_0$ then

- $\nu(B) \leq \mu(B)$ for all $B \in \mathcal{M}$
- $\nu(B) = \mu(B)$ if B is μ-σ-finite (if B is the countable union of finite sets w.r.t. μ)

This gives a good way to construct a measure

$$\underset{\text{premeasure}}{\mu_0} \rightsquigarrow \underset{\text{outer measure}}{\mu^\star} \rightsquigarrow \underset{\text{measure}}{\mu}$$

so that μ extends μ_0.

Proof.

(i) Let $A \in \mathcal{A}$. By definition $\mu^\star(A) \leq \mu_0(A)$. Now if $\{A_i\}_{i\in\mathbb{N}}$ is a collection of sets in $\mathcal{A}$ such that $\bigcup_{i=1}^\infty A_i \supset A$, let $B_i = A \cap \left(A_i \setminus \bigcup_{j=1}^{i-1} A_j\right)$. The B_i's are disjoint and have union A. So using the fact that a premeasure is countably additive and monotone:

$$\mu_0(A) = \sum_{i=1}^\infty \mu_0(B_i) \leq \sum_{i=1}^\infty \mu_0(A_i).$$

And since $\{A_i\}$ was arbitrary, we must have $\mu_0(A) \leq \mu^\star(A)$. Hence they are equal.

(ii) Carathéodory implies that $\mathcal{M}^\star$ is a σ-algebra and that $\mu = \mu^\star \restriction \mathcal{M}$ is a measure. Thus by minimality it suffices to show that $\mathcal{A} \subset \mathcal{M}^\star$. Let $A \in \mathcal{A}$, $E \subset X$, $\varepsilon > 0$. By definition of $\mu^\star$ there is a cover $\{B_i\}_{i\in\mathbb{N}}$ such that $E \subset \bigcup_{i=1}^\infty B_i$ and

$$\mu^\star(E) \geq \sum_{i=1}^\infty \mu_0(B_i) - \varepsilon = \sum_{i=1}^\infty (\mu_0(B_i \cap A) + \mu_0(B_i \cap A^c)) + \varepsilon \geq \mu^\star(E \cap A) + \mu^\star(E \cap A^c) + \varepsilon.$$

Above we use the fact that μ_0 is a premeasure. Then since $\varepsilon > 0$ was arbitrary we get that A is $\mu^\star$ measurable.

(iii) Let $B \in \mathcal{M}$, $B \subset \bigcup_{i=1}^\infty A_i$, with $A_i \in \mathcal{A}$. Then $\nu(B) \leq \sum_{i=1}^\infty \nu(A_i) = \sum_{i=1}^\infty \mu_0(A_i)$. Taking the infinum over all such covers yields $\nu(B) \leq \mu^\star(B) = \mu(B)$.

Now pick B such that $\mu(B) < \infty$. Let $\varepsilon > 0$. There exists a cover $\{A_i\}_{i\in\mathbb{N}}$ with $A_i \in \mathcal{A}$ and $B \subset \bigcup_{i=1}^\infty A_i := A$ such that $\sum_{i=1}^\infty \mu_0(A_i) \leq \mu^\star(B) + \varepsilon$. Then:

$$\mu(A) \leq \sum_{i=1}^\infty \mu(A_i) = \sum_{i=1}^\infty \mu_0(A_i) \leq \mu^\star(B) + \varepsilon = \mu(B) + \varepsilon.$$

Moreover, $B \subset A$ so $A = B \cup (A \cap B^c)$ and $\mu(A) = \mu(B) + \mu(A \cap B^c)$. Therefore $\mu(A \cap B^c) \leq \varepsilon$.

Finally:

$$\mu(B) \leq \mu(A) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} \mu_0\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} \nu\left(\bigcup_{i=1}^{n} A_i\right) = \nu(A) = \nu(B) + \underbrace{\nu(A \cap B^c)}_{\leq \mu(A\cap B^c) \leq \varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary we get $\mu(B) \leq \nu(B)$.

Now let B be μ-σ-finite. Namely $B = \bigcup_{i=1}^{\infty} B_i$ with $\mu(B_i) < \infty$. (Note that it suffices to show the result for disjoint B_i). Then

$$\mu(B) = \sum_{i=1}^{\infty} \mu(B_i) = \sum_{i=1}^{\infty} \nu(B_i) = \nu(B).$$

$\square$

1.4 Borel measures on the real line

Proposition 1.24. Let $F, G : \mathbb{R} \to \mathbb{R}$ be non-decreasing, right continuous functions. Then

(i) There is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$

(ii) $\mu_F = \mu_G$ if and only if $F - G$ is constant

(iii) If μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, then $\mu = \mu_F$ for

$$F(x) = \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0 \end{cases}$$

Proof.

(i) $\mu_0((a,b]) = F(b) - F(a)$ is a premeasure. Moreover $\mathcal{B}_{\mathbb{R}}$ is generated by $(a,b]$. By the previous theorem, μ_0 yields a unique Borel measure μ.

(ii)

$$\begin{aligned} \mu_F = \mu_G &\iff \text{equality of premeasures} \\ &\iff F(b) - F(a) = G(b) - G(a) \\ &\iff F \text{ and } G \text{ differ by a constant} \end{aligned}$$

(iii) Since μ is monotone, F is non-decreasing. Moreover for $a > 0$:

$$\lim_{n\to\infty} (F(a + 1/n) - F(a)) = \lim_{n\to\infty} (\mu(a, a + 1/n]) = \mu\left(\bigcap_{n=1}^{\infty} (a, a+1/n]\right) = \mu(\emptyset) = 0.$$

Hence F is right continuous (check the other cases). Finally

$$F(b) - F(a) = \begin{cases} \mu((0,b]) - \mu((0,a]) & 0 \leq a < b \\ \mu((0,b]) + \mu((a,0]) & a < 0 \leq b = \mu((a,b]) \\ -\mu((b,0]) + \mu((a,0]) & a < b < 0 \end{cases}$$

$\square$

Note 1.25.

(i) Every Borel measure on $\mathbb{R}$ that is finite on bounded Borel sets is of the form μ_F for some F. μ_F is called the **Lebesgue-Stieltjes** measure.

(ii) The case of $F(x) = x$ yields the **Lebesgue** measure denoted $m(E)$.

- It has domain $\mathcal{L} \supsetneq \mathcal{B}_{\mathbb{R}}$
- It is translation invariant: for $E \in \mathcal{L}$, $t \in \mathbb{R}$ we have $E + t \in \mathcal{L}$ and $m(E + t) = m(E)$

(iii) As an example consider $F(x) = x\mathbb{I}(x > 0)$. Then $\mu_F^\star(S) = 0$ for any $S \subset (-\infty, 0)$ and so any such set is measurable but not all are Borel.

Theorem 1.26. Let $\mu = \mu_F$ be a Lebesgue-Stieltjes measure and let $\mathcal{M}_\mu$ be its domain. Let $E \subset \mathcal{M}_\mu$. By definition:

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} \underbrace{(F(b_i) - F(a_i))}_{\mu((a_i,b_i])} : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i] \right\}.$$

We then have:

$$\begin{aligned} \mu(E) &\overset{(i)}{=} \inf \left\{ \sum_{i=1}^{\infty} \mu((a_i, b_i)) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i) \right\} \\ &\overset{(ii)}{=} \inf \{\mu(O) : E \subset O \text{ and } O \text{ is open}\} \\ &\overset{(iii)}{=} \sup \{\mu(K) : K \subset E \text{ and } K \text{ is compact}\} \end{aligned}$$

Proof.

(i) Call the R.H.S. $\nu(E)$.

- $\mu(E) \leq \nu(E)$:
  Let $a < b$ and choose an increasing sequence $\{c_i\}_{i\in\mathbb{N}}$ with $c_1 = a$, $c_i < b$ and $\lim\limits_{i\to\infty} c_i = b$. Then $(a, b) = \bigcup_{i=2}^{\infty}(c_{i-1}, c_i]$ and since they are disjoint $\mu((a, b)) = \sum_{i=2}^{\infty} \mu((c_{i-1}, c_i])$ so any countable sum of open intervals is a countable sum of half-open intervals. Thus by properties of infinum: $\mu(E) \leq \nu(E)$.
- $\mu(E) \geq \nu(E)$:
  Let $\varepsilon > 0$ and $E \subset \bigcup_{i=1}^{\infty}(a_i, b_i]$ be such that $\sum_{i=1}^{\infty} \mu((a_i, b_i]) \leq \mu(E) + \varepsilon$. By right continuity there exist $b_i' > b_i$ such that $F(b_i') < F(b_i) + \varepsilon/2^i$. Notice that
  $$\bigcup_{i=1}^{\infty}(a_i, b_i] \subset \bigcup_{i=1}^{\infty}(a_i, b_i').$$
  And so $\nu(E) \leq \sum_{i=1}^{\infty} F(b_i') - F(a_i) < \sum_{i=1}^{\infty}(F(b_i) - F(a_i)) + \varepsilon \leq \mu(E) + 2\varepsilon$.

(ii) Call the R.H.S. $\tilde{\nu}$.

- $\mu(E) \leq \tilde{\nu}(E)$:
  For any O with $E \subset O$, we have $\mu(E) \leq \mu(O)$ and the claim follows by taking the infinum over all such O

- $\mu(E) \geq \tilde{\nu}(E)$:
  Let $\varepsilon > 0$. By (i) there is a cover of open intervals $E \subset \bigcup_{i=1}^{\infty}(a_i, b_i)$ with

$$\sum_{i=1}^{\infty} \mu((a_i, b_i)) \leq \mu(E) + \varepsilon.$$

  Now since the countable union of open sets is open: $\tilde{\nu}(E) \leq \mu\left(\bigcup_{i=1}^{\infty}(a_i, b_i)\right) \leq \mu(E) + \varepsilon$. And since $\varepsilon > 0$ was arbitrary we obtain $\tilde{\nu}(E) \leq \mu(E)$.

(iii) Since for any $K \subset E$ we have $\mu(K) \leq \mu(E)$, taking the supremum over all such K yields

$$\mu(E) \geq \sup\{\mu(K) : K \subset E,\ K \text{ is compact}\}.$$

For the other inequality we split into two cases.

- If E is bounded, then $E \subset [-n, n]$ for some $n \in \mathbb{N}$. Then let $\varepsilon > 0$ and by (ii) we can choose an open set O such that $O \supset [-n, n] \setminus E$ and $\mu(O) \leq \mu([-n, n] \setminus E) + \varepsilon$. Define $K = [-n, n] \setminus O = [-n, n] \cap O^c$. Note that K is closed as the intersection of two closed sets and is bounded, so K is compact. Finally:

$$\mu(K) = \mu([-n, n]) - \mu(O) \geq \mu([-n, n]) - \mu([-n, n] \setminus E) - \varepsilon = \mu(E) - \varepsilon.$$

  Since $\varepsilon > 0$ was arbitrary we conclude that $\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ is compact}\}$.
- If E is unbounded, then define $E_n = E \cap [-n, n]$. E_n is bounded and so by the above there exists compact $K_n \subset E_n \subset E$ such that

$$\mu(E_n) - \frac{1}{n} \leq \mu(K_n) \leq \mu(E_n).$$

  So by the squeeze theorem: $\lim_{n\to\infty} \mu(K_n) = \mu(E)$. In particular for any $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $\mu(E) - \mu(K_{n_0}) \leq \varepsilon$. In particular $\mu(K_{n_0}) \geq \mu(E) - \varepsilon$. Since $\varepsilon > 0$ was arbitrary we conclude the desired result.

□

Corollary 1.27. Let $E \subset \mathbb{R}$. The following are equivalent

(i) $E \subset \mathcal{M}_\mu$

(ii) $E = V \setminus N$ where V is G_δ set and $\mu^\star(N) = 0$

(iii) $E = H \cup \tilde{N}$ where H is a countable union of compact sets and $\mu^\star(\tilde{N}) = 0$

Proof.

- (i) $\iff$ (ii):

  If V is G_δ set, then it is Borel and hence measurable. Moreover $\mu^\star(N) = 0$ implies that N is $\mu^\star$-measurable. So $E = V \setminus N = E \cap N^c \in \mathcal{M}_\mu$.

  Conversely, by the previous theorem for any $k \in \mathbb{Z}$ and $j \in \mathbb{N}$ there exists an open set $O_{j,k}$ such that $E \cap [k, k+1] \subset O_{j,k}$ and $\mu(O_{j,k}) \leq \mu(E \cap [k, k+1]) + 1/2^{j+|k|}$. Noting that $O_{j,k} =$

$(O_{j,k} \setminus (E \cap [k, k+1])) \cup (E \cap [k, k+1])$ we can conclude that $\mu(O_{j,k} \setminus (E \cap [k, k+1])) \leq 1/2^{j+|k|}$. Then $V = \bigcap_{j=1}^{\infty} (\bigcup_{k \in Z} O_{j,k})$ is a G_δ set such that $E \subset V$ (since $\bigcup_{k \in Z} O_{j,k}$ is open and covers E for any j). Then let $N = V \setminus E$. Notice that

$$\mu^\star(N) \leq \mu^\star \left(\left(\bigcup_{k \in \mathbb{Z}} O_{j,k} \right) \setminus \underbrace{E}_{\bigcup_{k \in \mathbb{Z}} (E \cap [k,k+1])} \right) \leq \sum_{k \in \mathbb{Z}} \mu \left(O_{j,k} \setminus (E \cap [k, k+1]) \right) \leq \sum_{k \in \mathbb{Z}} \frac{1}{2^{j+|k|}} = \frac{3}{2^j}.$$

(Note that $\sum_{k \in \mathbb{Z}} 2^{-|k|} = 1 + 2 \sum_{k=1}^{\infty} 2^{-k} = 1 + 2 = 3$). And since the above holds for any $j \in \mathbb{N}$, we have that $\mu^\star(N) = 0$. Finally this implies that $\mu(N) = 0$.

- (i) $\iff$ (iii)

  Suppose that $E = H \cup N$ where H is a countable union of compact sets and $\mu^\star(N) = 0$. Since compact sets are closed we know they are Borel and hence $\mu^\star$-measurable. Furthermore $\mu^\star(N) = 0$ implies that N is $\mu^\star$-measurable. Thus E is a countable union of $\mu^\star$-measurable sets and is hence $\mu^\star$-measurable.

  Conversely, suppose that E is $\mu^\star$-measurable. Then $E_k = E \cap [k, k+1]$ is $\mu^\star$-measurable and by (i) there exists compact $K_{j,k}$ such that $K_{j,k} \subset E_k$ and $\mu^\star(K_{j,k}) \geq \mu^\star(E_k) - 1/2^{j+|k|}$. Then since we can write $E_k = (E_k \setminus K_{j,k}) \cup K_{j,k}$ we have that

$$\mu^\star(E_k) \geq \mu^\star(E_k \setminus K_{j,k}) + \mu^\star(K_{j,k}) \geq \mu^\star(E_k \setminus K_{j,k}) + \mu^\star(E_k) - 1/2^{j+|k|}.$$

  In particular

$$\mu^\star(E_k \setminus K_{j,k}) \leq 1/2^{j+|k|}.$$

  Then $H = \bigcup_{j=1}^{\infty} \left(\bigcup_{k \in \mathbb{Z}} K_{j,k} \right)$ is a countable union of compact sets and $H \subset E$. Let $N = E \setminus H$. Then

$$\begin{aligned} \mu^\star(N) &= \mu^\star \left[\left(\bigcup_{k \in \mathbb{Z}} E_k \right) \setminus \left(\bigcup_{j=1}^{\infty} \bigcup_{k \in \mathbb{Z}} K_{j,k} \right) \right] = \mu^\star \left[\left(\bigcup_{k \in \mathbb{Z}} E_k \right) \cap \left(\bigcap_{j=1}^{\infty} \left(\bigcup_{k \in \mathbb{Z}} K_{j,k} \right)^c \right) \right] \\ &= \mu^\star \left[\bigcap_{j=1}^{\infty} \left(\bigcup_{k \in \mathbb{Z}} E_k \setminus \bigcup_{k \in \mathbb{Z}} K_{j,k} \right) \right] \leq \mu^\star \left[\bigcap_{j=1}^{\infty} \bigcup_{k \in \mathbb{Z}} (E_k \setminus K_{j,k}) \right] \end{aligned}$$

  Where this last inequality follows from $\bigcup A_k \setminus \bigcup B_k \subset \bigcup (A_k \setminus B_k)$ and the monotonicity of outer measure.

  Continuing we obtain:

$$\mu^\star(N) \leq \mu^\star \left(\bigcup_{k \in \mathbb{Z}} E_k \setminus K_{j,k} \right) \leq \sum_{k \in \mathbb{Z}} \mu^\star(E_k \setminus K_{j,k}) \leq \sum_{k \in \mathbb{Z}} \frac{1}{2^{j+|k|}} = \frac{3}{2^j}.$$

  Since this holds for all $j \in \mathbb{N}$ we conclude that $\mu^\star(N) = 0$ and therefore that $\mu(N) = 0$ completing the proof.

$\square$

1.5 The Cantor Set and the Cantor Function

Represent $x \in [0,1]$ by a ternary expansion $x = \sum_{n=0}^{\infty} x_n 3^{-n}$ with $x_n \in \{0,1,2\}$. To make this unique we avoid terminal 1's and keep terminal 2's. For example $1/3 = 0.022\dots$ rather than 0.1 and $2/3 = 0.2$ rather than $0.122\dots$. Or recursively: let $x_0 = 0$; and given $x_0, \dots, x_n$ let $\varepsilon_n = x - \sum_{j=0}^{n} x_j 3^{-j}$ and

$$x_{n+1} = \begin{cases} 0 & 0 \le \varepsilon_n \le 1/3^{n+1} \\ 1 & 1/3^{n+1} < \varepsilon_n < 2/3^{n+1} \\ 2 & 2/3^{n+1} \le \varepsilon_n \le 1/3^n \end{cases}.$$

Definition 1.28. The Cantor set C is $C = \{x \in [0,1] : x_n \neq 1 \text{ for all } n \in \mathbb{N}\}$.

In other words $C = [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \setminus \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)\right) \setminus \cdots$

Note 1.29.

- C is compact since it is the closed subset of a compact set
- C has empty interior, hence it is nowhere dense
- C is totally disconnected: if $\alpha < \beta$ in C were connected then $(\alpha, \beta) \subset C$ contradicting above
- C is uncountable (ternary expansion minus 1's)
- $m(C) = 0$. Indeed $m(C) = m([0,1]) - m((1/3, 2/3)) - \cdots = 1 - \sum_{n=0}^{\infty} \frac{2^n}{3n+1} = 1 - \frac{1}{3} \frac{1}{1 - \frac{2}{3}} = 0.$

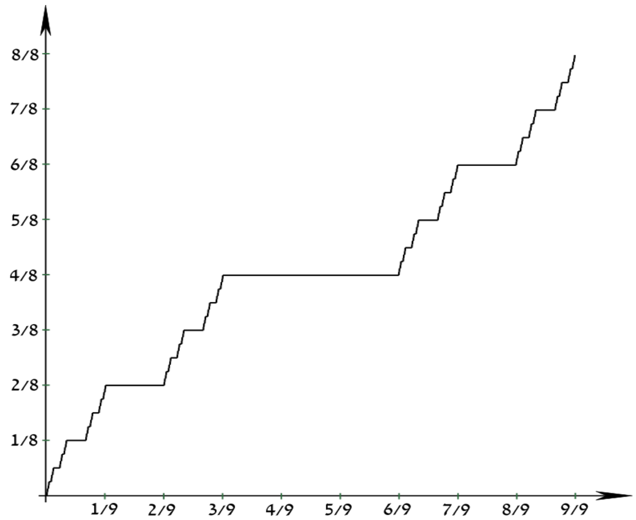
Note 1.30. Since $m(C) = 0$ and m is complete, we have that $\mathcal{P}(C) \subset \mathcal{L}$. In particular, since $\text{card}(C) = \text{card}(\mathbb{R})$, $\text{card}(\mathcal{L}) \geq \text{card}(\mathcal{P}(\mathbb{R})) > \text{card}(\mathbb{R})$. And since $\text{card}(\mathcal{B}(\mathbb{R})) = \text{card}(\mathbb{R})$ we conclude that $\mathcal{B}(\mathbb{R})$ is a strict subset of $\mathcal{L}$.

Definition 1.31. The Cantor function:

$$f : x = \sum_{n=0}^{\infty} x_n 3^{-n} \mapsto \begin{cases} \sum_{n=0}^{\infty} \frac{x_n}{2} 2^{-n} & x \in C \\ \sum_{n=0}^{N} \frac{x_n}{2} 2^{-n} + \frac{1}{2^{N+1}} & x \notin C, x_{N+1} = 1, x_n \in \{0,2\} \text{ for } n \leq N \end{cases}.$$

For elements in the Cantor set f maps to the corresponding binary representation, for elements not in the Cantor set f is piecewise constant. This function is

- non-decreasing
- continuous [f is onto $[0,1]$ but monotone functions can only have jump discontinuities]
- and constant almost everywhere [only increasing on C which has measure 0]



Proposition 1.32. Now we construct Vitali set. Define an equivalence relation on $[0,1)$ by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. Now $[0,1)$ is the disjoint union of equivalence classes $[x]$. Using the axiom of choice pick an element from each class to generate a set N. Then N is not measurable.

Proof. For any $r \in \mathbb{Q} \cap [0,1)$, let $\tilde{N}_r = N + r$ and then let $N_r = (\tilde{N}_r \cap [0,1)) \cup (\tilde{N}_r \cap [1,2) - 1)$. That is, shift N by any rational, and move whatever sticks off the end to front. Then

- $N_r \subset [0,1)$
- $N_r \cap N_s = \emptyset$ if $r \neq s$. Indeed if $y \in N_r \cap N_s$, then $y = x + r$ and $y = x' + s$. Namely $x - x' = r + s \in \mathbb{Q}$. So $x \sim x'$. But since $r \neq s$ we have $x \neq x'$. In particular N contains two different members of the same class, a contradiction.
- Assuming N is Lebesgue measurable:

  $$m(N_r) = m(\tilde{N}_r \cap [0,1)) + m(\tilde{N}_r \cap [1,2) - 1) = m(\tilde{N}_r \cap [0,1)) + m(\tilde{N}_r \cap [1,2)) = m(\tilde{N}_r) = m(N)$$

- $[0,1) \subset \bigcup_{r \in \mathbb{Q} \cap [0,1)} N_r$. Indeed if $y \in [0,1)$ then there is $x \in N$ such that $x \sim y$. Thus $y = x + r$ for some $r \in \mathbb{Q} \cap (-1,1)$. If $r \in [0,1)$ then $y \in N_r$ otherwise $y = (x + r + 1) - 1 \in N_{r+1}$.

Finally we can write

$$1 = m([0,1)) = \sum_r m(N_r) = \sum_r m(N).$$

Either $m(N) = 0$ or $m(N) > 0$ but both lead to a contradiction. □

Note 1.33. One may compute the outer measure of the Vitali set $m^\star(N) > 0$:

$$1 = m^\star([0,1)) \leq \sum_r m^\star(N_r) = \sum_r m^\star(N).$$

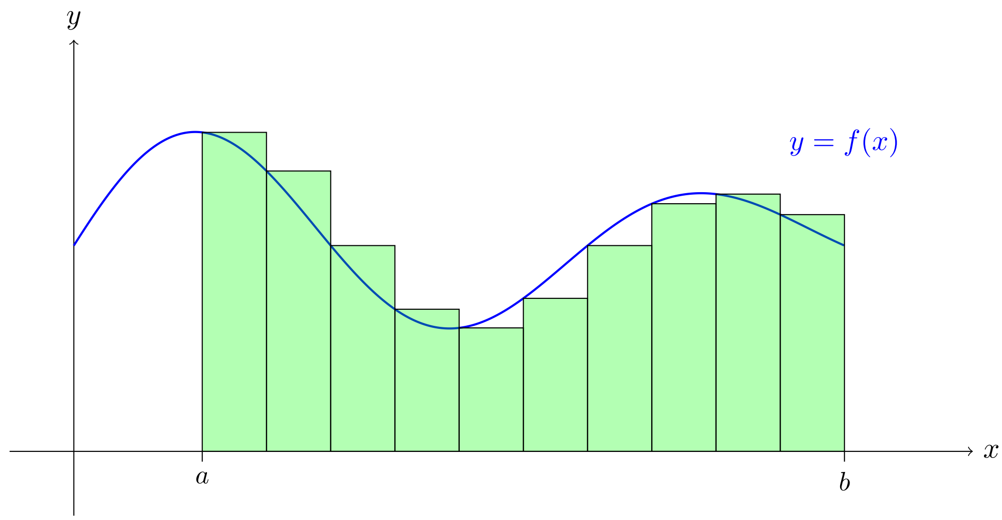
Noting that outer measures are only guaranteed to be subadditive even for disjoint sets.

2 Integration

We wish to define $\int_a^b f(x)d\mu(x)$.

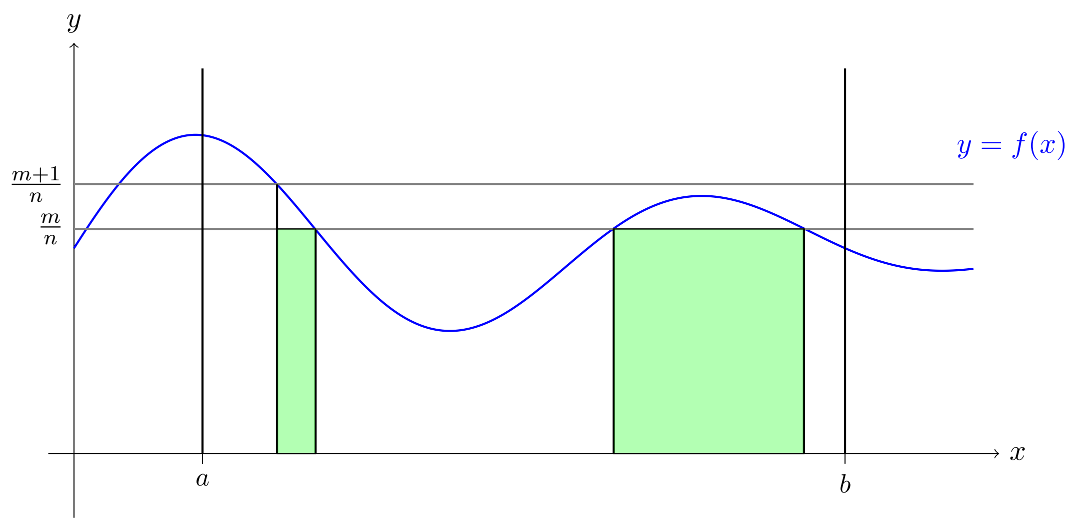
In the classical Riemann setting we slice the horizontal axis into finer and finer pieces. And then

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_m \frac{b-a}{n} f(x_m)$$



For the Lebesgue integral we will slice the vertical axis instead. And then

$$\int_a^b f(x)d\mu(x) = \lim_{n\to\infty}\sum_m \frac{m}{n}\mu\left(f^{-1}\left(\left[\frac{m}{n},\frac{m+1}{n}\right]\right)\cap[a,b]\right)$$



But in order to make sense of this, these pre-images must be measurable.

2.1 Measurable Functions

Definition 2.1. Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be measurable spaces. A function $f : X \to Y$ is said to be $(\mathcal{M}_X, \mathcal{M}_Y)$**-measurable** if $f^{-1}(E) \in \mathcal{M}_x$ for all $E \in \mathcal{M}_Y$.

- We say that $f : X \to \mathbb{R}$ is $\mathcal{M}_X$**-measurable** if f is $(\mathcal{M}_X, \mathcal{B}(\mathbb{R}))$-measurable
- A function $f : \mathbb{R} \to \mathbb{R}$ is called:
  (1) **Borel-measurable** if it is $(\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable
  (2) **Lebesgue-measurable** if it is $(\mathcal{L}, \mathcal{B}(\mathbb{R}))$-measurable
- This can be extended to functions $f : X \to \bar{\mathbb{R}}$ where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and

$$\mathcal{B}(\bar{\mathbb{R}}) = \{E \subset \bar{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}(\mathbb{R})\}$$

Lemma 2.2. Let $f : X \to Y$. Then

(i) $\{f^{-1}(A) : A \in \mathcal{M}_Y\} \subset \mathcal{P}(X)$ and $\{A : f^{-1}(A) \in \mathcal{M}_X\} \subset \mathcal{P}(Y)$ are σ-algebras

(ii) If $\mathcal{M}_Y$ is generated by $\mathcal{E}$, then f is $(\mathcal{M}_X, \mathcal{M}_Y)$-measurable if and only if $f^{-1}(E) \in \mathcal{M}_x$ for all $E \in \mathcal{E}$.

Proof.

(i) If $A \in f^{-1}(\mathcal{M}_Y)$, then there exist $N \in \mathcal{M}_Y$ such that $A = f^{-1}(N)$. Then $A^c = f^{-1}(N^c)$ so $A^c \in f^{-1}(\mathcal{M}_Y)$ since $N^c \in \mathcal{M}_Y$ (as $\mathcal{M}_Y$ is a σ-algebra). Similarly for a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $f^{-1}(\mathcal{M}_Y)$ there is $\{N_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}_Y$ such that $A_n = f^{-1}(N_n)$, and so

$$\bigcup_{n=1}^{n} A_n = f^{-1}\left(\bigcup_{n=1}^{\infty} N_n\right) \in f^{-1}(\mathcal{M}_Y).$$

Now let $\mathcal{M} = \{A \subset Y : f^{-1}(A) \in \mathcal{M}_X\} \subset \mathcal{P}(Y)$. Let $A \in \mathcal{M}$, we have that $f^{-1}(A) \in \mathcal{M}_X$. Thus $f^{-1}(A^c) = (f^{-1}(A))^c \in \mathcal{M}_X$ since $\mathcal{M}_X$ is a σ-algebra. Thus $A^c \in \mathcal{M}$. Similarly for a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}$ we have that $f^{-1}(A_n) \in \mathcal{M}_X$ and so

$$f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(A_n) \in \mathcal{M}_X.$$

Thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

(ii) If f is $(\mathcal{M}_X, \mathcal{M}_Y)$ measurable, then $\mathcal{E} \subset \mathcal{M}_Y$ and there is nothing to prove. Conversely, if $f^{-1}(E) \in \mathcal{M}_X$ for all $E \in \mathcal{E}$ then $\mathcal{M} = \{A \subset Y : f^{-1}(A) \in \mathcal{M}_X\}$ is a σ-algebra that contains $\mathcal{E}$. Since $\mathcal{M}_Y$ is the smallest σ-algebra containing $\mathcal{E}$, we conclude that $\mathcal{M}_Y \subset \mathcal{M}$ and so in particular $f^{-1}(A) \in \mathcal{M}_X$ for all $A \in \mathcal{M}_Y$ (that is f is $(\mathcal{M}_X, \mathcal{M}_Y)$-measurable).

$\square$

Note 2.3. The σ-algebra $\mathcal{M} := f^{-1}(\mathcal{M}_Y)$ is the smallest σ-algebra on X such that f is $(\mathcal{M}, \mathcal{M}_Y)$-measurable. If $\mathcal{M}'$ is another such σ-algebra then for each $A \in \mathcal{M}_Y$, $f^{-1}(A) \in \mathcal{M}'$. Namely $\mathcal{M} \subset \mathcal{M}'$. $\mathcal{M}$ is called the **σ-algebra generated by** f.

Corollary 2.4. If X, Y are topological spaces and $f : X \to Y$ is continuous. Then f is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable.

Proof. If $A \subset Y$ is open then by continuity of f so is $f^{-1}(A) \subset X$ open. Since the open sets in Y generate $\mathcal{B}(Y)$, and the open sets in X are measurable; we conclude by Lemma 2.2 (ii). $\square$

Corollary 2.5. Let $(X, \mathcal{M}_X)$ be a measurable space. Then $f : X \to \mathbb{R}$ is $\mathcal{M}_X$-measurable if and only if

(i) $f^{-1}((a, \infty)) \in \mathcal{M}_X$ for all $a \in \mathbb{R}$

(ii) $f^{-1}([a, \infty)) \in \mathcal{M}_X$ for all $a \in \mathbb{R}$

(iii) $f^{-1}((-\infty, a]) \in \mathcal{M}_X$ for all $a \in \mathbb{R}$

(iv) $f^{-1}((-\infty, a)) \in \mathcal{M}_X$ for all $a \in \mathbb{R}$

Proof. By **Lemma 2.1** (ii) and the fact that these intervals generate $\mathcal{B}(\mathbb{R})$. $\square$

Example 2.6.

- if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then it is Borel measurable
- if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then it is Borel measurable
- If $(X, \mathcal{M})$ is a measurable space, $E \in \mathcal{M}$, then the characteristic function χ_E is $\mathcal{M}$-measurable. Indeed $\chi_E^{-1}(A) = \begin{cases} X & \text{if } \{0,1\} \subset A \\ E & \text{if } 1 \in A,\ 0 \notin A \\ E^c & \text{if } 0 \in A,\ 1 \notin A \\ \emptyset & \text{otherwise} \end{cases} \in \mathcal{M}$.

  And if E is not measurable then χ_E is not $\mathcal{M}$-measurable since $\chi_E^{-1}((1/2, \infty)) = E \notin \mathcal{M}$.

Theorem 2.7. Let $(X, \mathcal{M})$ be a measurable space, $f, g : X \to \mathbb{R}$ be $\mathcal{M}$-measurable, and $c \in \mathbb{R}$. Then:

(i) $f + c$ and cf are $\mathcal{M}$-measurable

(ii) $f + g$ is $\mathcal{M}$-measurable

(iii) fg is $\mathcal{M}$-measurable

(iv) $\max\{f, g\}$ and $\min\{f, g\}$ are $\mathcal{M}$-measurable

(v) If $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of $\mathcal{M}$-measurable functions, then h is $\mathcal{M}$-measurable where:

   - $h(x) = \sup_n f_n(x)$
   - $h(x) = \inf_n f_n(x)$
   - $h(x) = \liminf_n f_n(x)$
   - $h(x) = \limsup_n f_n(x)$
   - if it exists: $h(x) = \lim_{n \to \infty} f_n(x)$

(vi) if $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $h \circ g : X \to \mathbb{R}$ is $\mathcal{M}$-measurable

Proof.

(i) $(f+c)^{-1}((a,\infty)) = f^{-1}((a-c,\infty)) \in \mathcal{M}$. If $c > 0$ then $(cf)^{-1}((a,\infty)) = f^{-1}((a/c,\infty)) \in \mathcal{M}$ and if $c < 0$ then $(cf)^{-1}((a,\infty)) = f^{-1}((-\infty, a/c)) \in \mathcal{M}$.

(ii) $(f+g)^{-1}((a,\infty)) = \{x \in X : f(x) + g(x) > a\} = \bigcup_{r\in\mathbb{Q}}\{x \in X : f(x) > r > a - g(x)\} = \bigcup_{r\in\mathbb{Q}} \left[f^{-1}((r,\infty)) \cap g^{-1}((a-r,\infty))\right]$.

(iii) $f(x)g(x) = \frac{1}{4}(f(x)+g(x))^2 - \frac{1}{4}(f(x)-g(x))^2$ so it suffices to check that f^2 is measurable.

$$(f^2)^{-1}((a,\infty)) = \begin{cases} f^{-1}((-\infty,-\sqrt{a})) \cup f^{-1}((\sqrt{a},\infty)) & a \geq 0 \\ X & a < 0 \end{cases} \in \mathcal{M}.$$

(iv) $\max\{f,g\}^{-1}((a,\infty)) = f^{-1}((a,\infty)) \cup g^{-1}((a,\infty)) \in \mathcal{M}$ and $\min\{f,g\} = -\max\{-f,-g\}$.

(v)
- $(\sup_n f_n)^{-1}((a,\infty)) = \bigcup_{n=1}^{\infty} f_n^{-1}((a,\infty)) \in \mathcal{M}$
- $\inf_n f_n = -\sup_n(-f_n)$
- $\liminf_n f_n = \sup_n \inf_{m\geq n} f_m$
- $\limsup_n f_n = \inf_n \sup_{m\geq n} f_m$
- If $\lim_{n\to\infty} f_n$ exists, then it is equal to $\limsup_n f_n$

(vi) $(h\circ g)^{-1}((a,\infty)) = \{x \in X : h\circ g(x) > a\} = \{x \in X : g(x) \in h^{-1}((a,\infty))\} = g^{-1}(h^{-1}((a,\infty)))$ which is in $\mathcal{M}$.

□

Definition 2.8. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f, g, \{f_n\}_{n\in\mathbb{N}} : X \to \mathbb{R}$ then we say

(i) $f = g$ a.e. ("almost everywhere") if there is a null set $E \in \mathcal{M}$ such that $f(x) = g(x)\ \forall x \in E^c$

(ii) $f = \lim_{n\to\infty} f_n$ a.e. of there is $E \in \mathcal{M}$, $\mu(E) = 0$ such that $\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in E^c$.

Lemma 2.9. Let $(X, \mathcal{M}, \mu)$ be a complete measure space.

(i) If f is measurable and $g = f$ a.e., then g is measurable

(ii) If $\{f_n\}_{n\in\mathbb{N}}$ are measurable for all $n \in \mathbb{N}$ and $f = \lim_{n\to\infty} f_n$ a.e. then f is measurable

Note 2.10. The completeness assumption in the above lemma is necessary. Let $(X, \mathcal{M}, \mu)$ be a measure space which is not complete. Pick a null set $N \in \mathcal{M}$ and a subset $Z \subset N$ such that $Z \notin \mathcal{M}$. Let $f(x) = 1$ for all $x \in X$ and $g(x) = \begin{cases} 1 & x \in X \setminus Z \\ 0 & x \in Z \end{cases}$. Then f is measurable, $f = g$ a.e. since $f(x) = g(x)$ for all $x \in X \setminus N$, but g is not measurable since $Z = g^{-1}((-\infty, 1/2)) \notin \mathcal{M}$.

2.2 Integration

Definition 2.11. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $E \in \mathcal{M}$. Let

$$L^+(X, \mathcal{M}) = \{f : X \to [0, \infty] : \ f \text{ is measurable}\}.$$

(i) For $f \in L^+(X, \mathcal{M})$:

$$\int_E f d\mu = \sup \left\{ \sum_{\substack{1 \le i \le n \\ a_i \ne 0}} a_i \mu(E_i \cap E) : \sum_{i=1}^n a_i \chi_{E_i}(x) \le f(x) \text{ for all } x \in X, 0 \le a_i < \infty, E_i \in \mathcal{M}, n \in \mathbb{N} \right\}.$$

(ii) $L^1(E, X, \mathcal{M}, \mu) = \left\{ f : X \to \mathbb{R}, \text{ such that } f \text{ is } \mathcal{M}\text{-measurable and } \int_E |f| d\mu < \infty \right\}$

(iii) For $f \in L^1(E, X, \mathcal{M}, \mu)$:

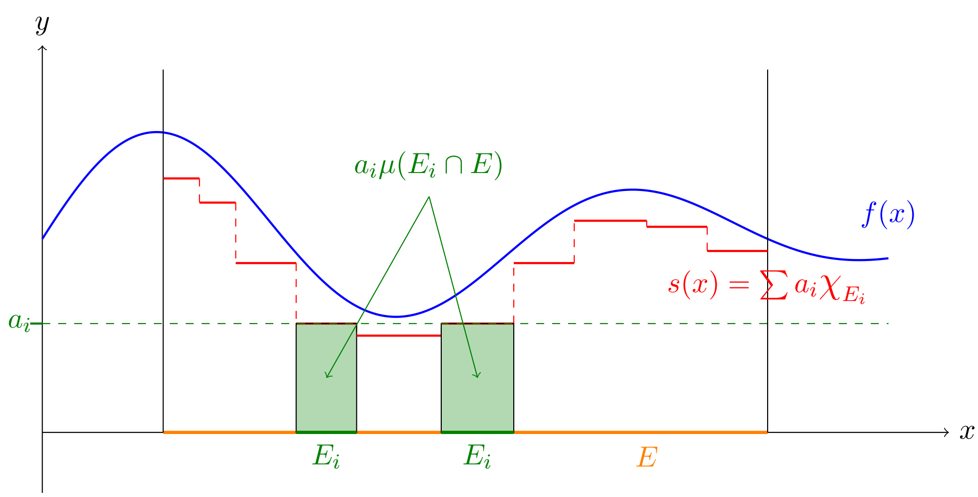
$$\int_E f d\mu = \int_E \max\{f, 0\} d\mu - \int_E \max\{-f, 0\} d\mu.$$

Note 2.12.

(i) The explicit $a_i \ne 0$ makes it clear that if f is identically zero on $E' \subset X$, then E' contributes zero to the integral even if $\mu(E') = \infty$

(ii) Concretely, a useful picture to have in mind is to let $a = \inf\{f(x) : x \in E\}$ and $a_i = a + \frac{i-1}{N}$; $E_i = f^{-1}\left(\left[a + \frac{i-1}{N}, a + \frac{i}{N}\right]\right)$ for $i \in \mathbb{N}$ and then let $N \to \infty$

(iii) Since $|f| = \max\{f, -f\}$ it is measurable. The same holds for $\max\{f, 0\}$ and $\max\{-f, 0\}$



Theorem 2.13. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $E \in \mathcal{M}$.

(i) If either $a_i \ge$ and $E_i \in \mathcal{M}$, or $a_i \in \mathbb{R}$ and $\mu(E_i \cap E) < \infty$, then

$$\int_E \left(\sum_{i=1}^n a_i \chi_{E_i} \right) d\mu = \sum_{\substack{1 \le i \le n \\ a_i \ne 0}} a_i \mu(E_i \cap E).$$

(ii) If $f, g \in L^1(E, X, \mathcal{M}, \mu)$, then $f + g \in L^1(E, X, \mathcal{M}, \mu)$ and

$$\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu.$$

(iii) If $f \in L^1(E, X, \mathcal{M}, \mu)$ and $\lambda \in \mathbb{R}$, then $\lambda f \in L^1(E, X, \mathcal{M}, \mu)$ and

$$\int_E \lambda f d\mu = \lambda \int_E f d\mu.$$

(iv) If $f \in L^1(E, X, \mathcal{M}, \mu)$, then $f\chi_E \in L^1(X, X, \mathcal{M}, \mu)$ and

$$\int_X (f\chi_E) d\mu = \int_E f d\mu.$$

Proof.

(i)
- Case 1: $a_i \geq 0$ for $1 \leq i \leq n$. By definition

$$\int_E \left(\sum_{i=1}^n a_i \chi_{E_i} \right) d\mu = \sup \left\{ \sum_{\substack{1 \leq j \leq m \\ b_j \neq 0}} b_j \mu(F_j \cap E) : \sum_{j=1}^m b_j \chi_{F_j} \leq \sum_{i=1}^n a_i \chi_{E_i} \right\}.$$

So we immediately have that

$$\int_E \left(\sum_{i=1}^n a_i \chi_{E_i} \right) d\mu \geq \sum_{\substack{1 \leq i \leq n \\ a_i \neq 0}} a_i \mu(E_i \cap E).$$

It suffices to show

$$\sum_{j=1}^m b_j \chi_{F_j} \leq \sum_{i=1}^n a_i \chi_{E_i} \implies \sum_{\substack{1 \leq j \leq m \\ b_j \neq 0}} b_j \mu(F_j \cap E) \leq \sum_{\substack{1 \leq i \leq n \\ a_i \neq 0}} a_i \mu(E_i \cap E).$$

Given $G_1, \ldots, G_p$ there exists pairwise disjoint $H_1, \ldots, H_q$ such that

$$G_k = \bigcup_{\substack{1 \leq j \leq q \\ H_j \subset G_k}} H_j.$$

Moreover

$$\sum_{k=1}^p c_k \chi_{G_k} = \sum_{\substack{j,k \\ H_j \subset G_k}} c_k \chi_{H_j} = \sum_{j=1}^q d_j \chi_{H_j}, \quad \text{where } d_j = \sum_{\substack{k \\ H_j \subset G_k}} c_k.$$

And similarly

$$\sum_{k=1}^p c_k \mu(G_k \cap E) = \sum_{j=1}^q d_j \mu(H_j \cap E).$$

So we can assume that $E_i = F_i$ and that these sets are pairwise disjoint by taking the G_k's above to be the whole family of E_i's and F_j's. Finally if $\sum_i b_i \chi_{E_i}(x) \leq \sum_i a_i \chi_{E_i}(x)$, then $b_i \leq a_i$ by taking $x \in E_i$. And so finally

$$\sum_i b_i \mu(E_i \cap E) \leq \sum_i a_i \mu(E_i \cap E).$$

- Case 2: $a_i \in \mathbb{R}$ and $\mu(E_i \cap E) < \infty$. Without loss of generality we can assume that the E_i's are disjoint. Then for $f = \sum_i a_i \chi_{E_i}$ we have

$$\int_E |f| d\mu = \int_E \sum_i |a_i| \chi_{E_i} d\mu = \sum_i |a_i| \mu(E_i \cap E) < \infty.$$

Thus $f \in L^1(E, X, \mathcal{M}, \mu)$. Furthermore:

$$\int_E f d\mu = \int_E \sum_{\substack{1 \leq i \leq n \\ a_i > 0}} a_i \chi_{E_i} d\mu - \sum_{\substack{1 \leq i \leq n \\ a_i < 0}} (-a_i) \chi_{E_i} d\mu = \sum_i a_i \mu(E_i \cap E).$$

(ii) ⚠ patience

(iii) • Case 1: $\lambda > 0$ and $f \geq 0$.

$$\begin{aligned}
\int_E (\lambda f) d\mu &= \sup \left\{ \sum_i a_i \mu(E_i \cap E) : \sum_i a_i \chi_{E_i} \leq \lambda f \right\} \\
&= \sup \left\{ \sum_i \lambda \tilde{a}_i \mu(E_i \cap E) : \sum_i \lambda \tilde{a}_i \chi_{E_i} \leq \lambda f \right\} \\
&= \lambda \sup \left\{ \sum_i \tilde{a}_i \mu(E_i \cap E) : \sum_i \tilde{a}_i \chi_{E_i} \leq f \right\} \\
&= \lambda \int_E f d\mu
\end{aligned}$$

- Case 2: general signs $\lambda \neq 0$. We have $|\lambda f| = |\lambda||f|$, so if $f \in L^1$ then by the above case

$$\int_E |\lambda f| d\mu = |\lambda| \int_E |f| d\mu < \infty,$$

and therefore $\lambda f \in L^1$ also. Now

$$\begin{aligned}
\int_E (\lambda f) d\mu &= \int_E \max\{|\lambda| \operatorname{sgn}(\lambda) f, 0\} d\mu - \int_E \max\{-|\lambda| \operatorname{sgn}(\lambda) f, 0\} d\mu \\
&= |\lambda| \int_E \max\{\operatorname{sgn}(\lambda) f, 0\} d\mu - |\lambda| \int_E \max\{-\operatorname{sgn}(\lambda) f, 0\} d\mu \\
&= \lambda \int_E \max\{f, 0\} d\mu - \lambda \int_E \max\{-f, 0\} d\mu \\
&= \lambda \int_E f d\mu
\end{aligned}$$

(iv) We claim that $S_1 = S_2$ where S_1 and S_2 are defined below as

$$S_1 = \left\{ \sum_i a_i \mu(E_i \cap E) : \sum_i a_i \chi_{E_i} \leq f \right\} \qquad \text{and} \qquad S_2 = \left\{ \sum_i a_i \mu(E_i) : \sum_i a_i \chi_{E_i} \leq f \chi_E \right\}.$$

If $\sum_i a_i \chi_{E_i}(x) \leq f(x)\chi_E(x)$, then either $E_i \subset E$ or $a_i = 0$. Therefore

$$S_2 = \left\{ \sum_{a_i \neq 0} a_i \mu(E_i \cap E) : \sum_i a_i \chi_{E_i} \leq f \chi_E \right\},$$

and so $S_2 \subset S_1$. On the other hand, pick any $\sum_i a_i \mu(E_i \cap E) \in S_1$, then $\sum_i a_i \chi_{E_i} \leq f$. Multiplying by χ_E yields $\sum_i a_i \chi_{E_i \cap E} \leq f \chi_E$. And so by the same observation as before: $\sum_i a_i \mu(E_i \cap E) \in S_2$. Thus

$$\int_X f \chi_E d\mu = \sup S_2 = \sup S_1 = \int_E f d\mu.$$

$\square$

Theorem 2.14. Let $(X, \mathcal{M}, \mu)$ be a measure space, $E \in \mathcal{M}$, $f, g, h : X \to \mathbb{R}$ measurable, and $f, g \in L^1(E, X, \mathcal{M}, \mu)$.

(i) If $|h| \leq f$ on E, then $h \in L^1(E, X, \mathcal{M}, \mu)$ and $\displaystyle\int_E h d\mu \leq \int_E f d\mu$.

(ii) $\displaystyle\left| \int_E f d\mu \right| \leq \int_E |f| d\mu$.

(iii) If $f \leq g$, then $\displaystyle\int_E f d\mu \leq \int_E g d\mu$.

(iv) If h is bounded and $\mu(E) < \infty$, then $h \in L^1(E, X, \mathcal{M}, \mu)$ and

$$\left| \int_E h d\mu \right| \leq \mu(E) \sup_{x \in E} |h(x)|.$$

Proof.

(i) We write

$$\begin{aligned}
\int_E |h| d\mu &= \int_X |h| \chi_E d\mu \\
&= \sup \left\{ \int_X \varphi d\mu : \varphi \text{ simple}, \varphi \leq |h| \chi_E \right\} \\
&\leq \sup \left\{ \int_X \varphi d\mu : \varphi \text{ simple}, \varphi \leq f \chi_E \right\} \\
&= \int_X f \chi_E d\mu \\
&= \int_E f d\mu.
\end{aligned}$$

(ii) Apply (i) with $h \mapsto \max\{\pm f, 0\}$ and $f \mapsto |f| = \max\{f,0\} + \max\{-f,0\}$. Then we have

$$\begin{aligned}\left|\int_E f d\mu\right| &= \left|\int_E \max\{f,0\}d\mu - \int_E \max\{-f,0\}d\mu\right| \\ &\le \max\left\{\int_E \max\{f,0\}d\mu, \int_E \max\{-f,0\}d\mu\right\} \\ &\overset{(i)}{\le} \int_E \max\{f,0\} + \max\{-f,0\}d\mu \\ &= \int_E |f|d\mu\end{aligned}$$

(iii) ⚠ using Theorem 2.13 (ii). If $f \le g$, then $g - f \ge 0$, therefore

$$\int_E g d\mu - \int_E f d\mu = \int (g-f)d\mu \ge 0.$$

(iv) Let $f = \left(\sup_{x\in E}|h(x)|\right)\chi_E$. Then f is simple so $\int_E f d\mu = \left(\sup_{x\in E}|h(x)|\right)\mu(E)$, so $f \in L^1$. Furthermore, $|h| \le f$, so by (i) $h \in L^1$ and by (ii)

$$\left|\int_E h d\mu\right| \le \int_E |h| d\mu \le \int_E f d\mu = \left(\sup_{x\in E}|h(x)|\right)\mu(E).$$

□

Note 2.15. Theorem 2.13 (ii) and Theorem 2.14 (iii) are not yet proved. However if $0 \le f \le g$:

$$\begin{aligned}\int_E f d\mu &= \sup\left\{\int_E \varphi d\mu : \varphi \text{ simple}, \varphi \le f\right\} \\ &\le \sup\left\{\int_E \varphi d\mu : \varphi \text{ simple}, \varphi \le g\right\} = \int_E g d\mu\end{aligned}$$

2.3 Limit Theorems

Lemma 2.16. *(Fatou's Lemma)* Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Let $f_n : X \to [0,\infty]$ be measurable for all $n \in \mathbb{N}$. Then

$$\int_E \liminf_{n\to\infty} f_n d\mu \le \liminf_{n\to\infty} \int_E f_n d\mu.$$

Note 2.17.

(i) For $g : X \to [0,\infty]$, g is measurable if and only if $g^{-1}((a,\infty]) \in \mathcal{M}$ for all $a \in \mathbb{R}$ since $(a,\infty]$ generates $\mathcal{B}(\bar{\mathbb{R}})$. Indeed $(a,b] = (a,\infty] \cap (b,\infty]^c$, $\{\infty\} = \bigcap_{n\in\mathbb{N}}(n,\infty]$ and $\{-\infty\} = \left(\bigcup_{n\in\mathbb{N}}(-n,\infty]\right)^c$

(ii) As an example, let

$$f_n(x) = \begin{cases} 1 & \text{if } |x| > n \\ 0 & \text{if } |x| \le n \end{cases}.$$

Then $\lim_{n\to\infty} f_n(x) = 0$ but $\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)dx = \infty$ which shows strict inequality is possible.

(iii) The definition of the Lebesgue Integral is the same so $\liminf_n f_n$ exists and is measurable.

Proof. We must show that

$$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x) \le \liminf_{n\to\infty} f_n(x) \implies \int_E \varphi d\mu \le \liminf_{n\to\infty} \int_E f_n d\mu.$$

- If $\int_E \varphi d\mu = \infty$, then there is an i such that $a_i > 0$ and $\mu(E_i \cap E) = \infty$. Thus for $x \in E_i \cap E$,

  $$\liminf_{n\to\infty} f_n(x) \ge a_i.$$

  In particular for all $\varepsilon > 0$ there exists n such that $f_k(x) \ge a_i - \varepsilon$ for all $k \ge n$, and by taking $\varepsilon = a_i/2$ there exists some n such that if $k \ge n$, then $f_k(x) \ge a_i/2$. Let

  $$A_n = \left\{x \in E : f_k(x) \ge \frac{a_i}{2},\ \text{for all } k \ge n\right\} = \left(\bigcap_{k\ge n} f_k^{-1}\left(\left[\frac{a_i}{2}, \infty\right]\right)\right) \cap E \in \mathcal{M}.$$

  Then $A_n \subset A_{n+1}$, $E_i \cap E \subset \bigcup_n A_n$, and $\int_E f_n d\mu \ge \int_E \frac{a_i}{2}\chi_{A_n} d\mu$ for all $n \in \mathbb{N}$. Hence (by continuity from below):

  $$\liminf_{n\to\infty} \int_E f d\mu \ge \liminf_{n\to\infty} \frac{a_i}{2}\mu(A_n) = \frac{a_i}{2}\mu\left(\bigcup_n A_n\right) \ge \frac{a_i}{2}\mu(E_i \cap E) = \infty.$$

- If $\int_E \varphi d\mu < \infty$, let $\varepsilon > 0$ and $c = 1 - \varepsilon$. Furthermore, define:

  $$A = \{x \in E : \varphi(x) > 0\} \qquad \text{and} \qquad A_n = \{x \in A : f_k(x) \ge c\varphi(x) \text{ for all } k \ge n\}.$$

  Then $A_n \subset A_{n+1}$, $A = \bigcup_n A_n \subset E$, and $\mu(A) < \infty$. By continuity from below $\mu(A) = \lim_n \mu(A_n)$, and since $\mu(A) = \mu(A_n) + \mu(A \setminus A_n)$ there is $N \in \mathbb{N}$ such that $\mu(A \setminus A_n) < \varepsilon$ for all $n \ge N$. Hence

  $$\begin{aligned}\int_E f_n d\mu &\ge \int_{A_n} f_n d\mu \\ &\ge \int_{A_n} c\varphi d\mu \\ &= \int_A c\varphi d\mu - c\sum_i a_i \underbrace{\mu(E_i \cap (A \setminus A_n))}_{<\varepsilon \text{ if } n \ge N} \\ &\ge \int_A \varphi d\mu - \varepsilon \int_A \varphi d\mu - \varepsilon c \sum_i a_i\end{aligned}$$

  This implies that

  $$\liminf_{n\to\infty} \int_E f_n d\mu \ge \int_A \varphi d\mu - \varepsilon\left(\int_A \varphi d\mu + c\sum_i a_i\right).$$

  And since this holds for all $\varepsilon > 0$ we have

  $$\liminf_{n\to\infty} \int_E f_n d\mu \ge \int_A \varphi d\mu = \int_E \varphi d\mu.$$

∎

Lemma 2.18. Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Let $f_n : X \to [0, \infty]$ be measurable for all $n \in \mathbb{N}$ and let

$$f(x) = \liminf_{n \to \infty} f_n \qquad \text{a.e. on } E.$$

Assume that f is measurable (this is automatic if μ is complete). Then

$$\int_E f d\mu \leq \liminf_{n \to \infty} \int_E f_n d\mu.$$

Proof. Let $\Omega \in \mathcal{M}$ be the null set such that $f(x) = \liminf_{n \to \infty} f_n(x)$ for all $x \in \Omega^c$. Let $g = f\chi_{\Omega^c}$ and $g_n = f_n\chi_{\Omega^c}$ for all $n \in \mathbb{N}$. Then $\int_E f d\mu = \int_E g d\mu$ since $f = g$ a.e. and the same is true for all f_n and g_n the proof of this is an exercise. The claim now follows from Fatou's Lemma. $\square$

Theorem 2.19. *(Monotone Convergence)* Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Let $f_n \in L^+(X, \mathcal{M})$ for all $n \in \mathbb{N}$ and $f : X \to (0, \infty)$ be such that $f(x) = \lim_n f_n(x)$ a.e. on E and $f_n(x) \leq f(x)$ a.e. on E for all $n \in \mathbb{N}$. If f is measurable, then

$$\int_E f d\mu = \lim_{n \to \infty} \int_E f_n d\mu.$$

Note that the assumptions are satisfied if $f_{n+1} \leq f_n$ for all $n \in \mathbb{N}$ and $f = \lim_n f_n$.

Proof. Since $0 \leq f_n \leq f$ a.e. we have

$$\int_E f_n d\mu \leq \int_E f d\mu.$$

And so by Fatou's Lemma:

$$\int_E f d\mu \leq \liminf_n \int_E f_n d\mu \leq \limsup_n \int f_n d\mu \leq \int_E f d\mu.$$

So all inequalities are equalities and in particular the limit of the integral exists and equals $\int_E f d\mu$. $\square$

Note 2.20. We are now ready to prove Theorem 2.13 (ii). Namely if $f, g \in L^1(E, X, \mathcal{M}, \mu)$, then $f + g \in L^1(E, X, \mathcal{M}, \mu)$ and

$$\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu.$$

- Case: f, g non-negative, simple, measurable. Let

$$f = \sum_i a_i \chi_{E_i} \qquad \text{and} \qquad g = \sum_i b_i \chi_{F_i}.$$

Without loss of generality we can assume $E_i = F_i$ and that they are pairwise disjoint. The claim then follows immediately from writing $f + g$ as a simple function and using

$$\int_E \left(\sum_i a_i \chi_{E_i} \right) d\mu = \sum_i a_i \mu(E_i \cap E).$$

- Case: $f, g \in L^+$. Let $I_{i,n} = \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)$ then define

$$f_n(x) = \sum_{i=0}^{4^n} \frac{i}{2^n} \chi_{f^{-1}(I_{i,n})} + 2^n \chi_{f^{-1}(\{\infty\})},$$

and similarly for g. Then $\{f_n\}$ and $\{g_n\}$ are non-negative, simple, and measurable. Furthermore $0 \leq f_n \leq f(x)$ and $f(x) = \lim_{n\to\infty} f_n(x)$. Thus by Monotone Convergence:

$$\int_E (f+g)d\mu = \lim_{n\to\infty} \int_E (f_n + g_n)d\mu = \lim_{n\to\infty} \left[\int_E f_n d\mu + \int_E g_n d\mu\right] = \int_E f d\mu + \int_E g d\mu.$$

- Case: $f, g \in L^1$ with $f + g \geq 0$ and $f \geq 0$. By the above:

$$\int_E |f| + |g| d\mu = \int_E |f| d\mu + \int_E |g| d\mu < \infty,$$

hence $|f| + |g| \in L^1$. Furthermore $|f + g| \leq |f| + |g|$ so $f + g \in L^1$. Define $g^+ = \max\{g, 0\}$ and $g^- = \max\{-g, 0\}$ so that $g = g^+ - g^-$. Then

$$\int_E g^+ d\mu + \int_E f d\mu = \int_E (g^+ + f)d\mu = \int_E [(f + g^+ - g^-) + g^-]d\mu = \int_E (f+g)d\mu + \int_E g^- d\mu.$$

And so

$$\int_E (f+g)d\mu = \int_E f d\mu + \left(\int_E g^+ d\mu - \int_E g^- d\mu\right) = \int_E f d\mu + \int_E g d\mu,$$

where the last inequality is the definition of $\int_E g d\mu$.

- General case: $f, g \in L^1 \implies f + g \in L^1$ as before. Thus using the fact that $|a| + a \geq 0$:

$$\begin{aligned}\int_E (|f| + |g|)d\mu + \int_E (f+g)d\mu &= \int_E (|f| + |g| + f + g)d\mu = \int_E (|f| + |g| + f)d\mu + \int_E g d\mu \\ &= \int_E (|f| + |g|)d\mu + \int_E f d\mu + \int_E g d\mu\end{aligned}$$

Theorem 2.21. *(Dominated Convergence Theorem)* Let $(X, \mathcal{M}, \mu)$ be a measure space, $E \in \mathcal{M}$, and $f, g : X \to \mathbb{R}$ be measurable. Furthermore let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions and $g \in L^1(E, X, \mathcal{M}, \mu)$. Suppose $f(x) = \lim_n f_n(x)$ a.e. on E and that $|f_n(x)| \leq g(x)$ a.e. on E for all $n \in \mathbb{N}$. Then $f \in L^1(E, X, \mathcal{M}, \mu)$ and

$$\int_E f d\mu = \lim_n \int_E f_n d\mu.$$

Note 2.22. The uniform boundedness is necessary. For example let

$$f_n(x) = \begin{cases} 1/n & |x| < \frac{n}{2} \\ 0 & \text{o.w.} \end{cases}.$$

Then $f(x) = \lim_n f_n(x) = 0$ a.e. and yet $\int_{\mathbb{R}} f_n(x) d\mu = 1$ for all $n \in \mathbb{N}$.

Proof. By Fatou's Lemma

$$\int_E g d\mu + \int_E f d\mu = \int_E (g+f) d\mu \leq \liminf_n \int_E (g+f_n) d\mu = \int_E g d\mu + \liminf_n \int_E f_n d\mu,$$

and therefore

$$\int_E f d\mu \leq \liminf_n \int_E f_n d\mu.$$

Repeat the above with $g - f$ to obtain

$$-\int_E f d\mu \leq -\limsup_n \int f_n d\mu.$$

Altogether we have:

$$\int_E f d\mu \leq \liminf_n \int f_n d\mu \leq \limsup_n \int_E f_n d\mu \leq \int_E f d\mu.$$

Hence all inequalities are equalities so the limit exists and is equal to $\int_E f d\mu$. $\square$

2.4 Riemann Integrals

We will now establish a relationship between the Lebesgue and Riemann integral.

Definition 2.23. Let $-\infty < a < b < \infty$ and let $f : [a, b] \to \mathbb{R}$ be bounded. Then

(i) The **upper Riemann integral** of f is

$$\overline{\int_a^b} f(x)dx = \inf \left\{ \sum_{i=1}^n (t_i - t_{i-1}) \sup_{t_{i-1} \leq x \leq t_i} f(x) : n \in \mathbb{N}, a = t_0 < t_1 < \cdots < t_n = b \right\}$$

(ii) The **lower Riemann integral** of f is

$$\underline{\int_a^b} f(x)dx = \sup \left\{ \sum_{i=1}^n (t_i - t_{i-1}) \inf_{t_{i-1} \leq x \leq t_i} f(x) : n \in \mathbb{N}, a = t_0 < t_1 < \cdots < t_n = b \right\}$$

(iii) The function f is **Riemann integrable** if

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx,$$

and in this case we denote the common value $\mathcal{R}\!\!\int_a^b f(x)dx$.

Note 2.24. If f is not bounded it cannot be Riemann integrable. For example if f is not bounded above then $\sum_i (t_i - t_{i-1}) \sup f(x) = \infty$ for every partition.

Theorem 2.25. Let $-\infty < a < b < \infty$ and let $f : [a, b] \to \mathbb{R}$ be bounded.

(i) If f is Riemann integrable, then $f \in L^1([a, b], \mathbb{R}, \mathcal{L}, m)$ and

$$\mathcal{R}\!\!\int_a^b f(x)dx = \int_{[a,b]} f(x)dm.$$

(ii) f is Riemann integrable if and only if $\{x \in [a,b] : f \text{ is not continuous at } x\}$ has Lebesgue measure zero.

Proof. We prove (i) and provide a sketch of the proof for (ii). First some notation:

- ⋆ $\mathbb{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$
- ⋆ $M_i = \sup\{f(x) : t_{i-1} \leq x \leq t_i\}$
- ⋆ $m_i = \inf\{f(x) : t_{i-1} \leq x \leq t_i\}$
- ⋆ $\overline{S_{\mathbb{P}}} = \sum_i (t_i - t_{i-1}) M_i$; and $\underline{S_{\mathbb{P}}} = \sum_i (t_i - t_{i-1}) m_i$
- ⋆ $\overline{g_{\mathbb{P}}} = \sum_i M_i \chi_{[t_{i-1}, t_i]}$; and $\underline{g_{\mathbb{P}}} = \sum_i m_i \chi_{[t_{i-1}, t_i]}$

We have that $\overline{g_{\mathbb{P}}} \geq f \geq \underline{g_{\mathbb{P}}}$ and

$$\int_{[a,b]} \overline{g_{\mathbb{P}}} dm = \overline{S_{\mathbb{P}}}; \qquad \int_{[a,b]} \underline{g_{\mathbb{P}}} dm = \underline{S_{\mathbb{P}}}.$$

Also if $\mathbb{P}'$ is a refinement of $\mathbb{P}$ (namely $\mathbb{P} \subset \mathbb{P}'$), then

$$\underline{S_{\mathbb{P}}} \leq \underline{S_{\mathbb{P}'}} \leq \overline{S_{\mathbb{P}'}} \leq \overline{S_{\mathbb{P}}}; \qquad \text{and} \qquad \underline{g_{\mathbb{P}}} \leq \underline{g_{\mathbb{P}'}} \leq \overline{g_{\mathbb{P}'}} \leq \overline{g_{\mathbb{P}}}.$$

Now since f is Riemann integrable we have

$$\inf_{\mathbb{P}} \overline{S_{\mathbb{P}}} = \sup_{\mathbb{P}} \underline{S_{\mathbb{P}}}.$$

By definition of infimum there is a sequence of partitions there is a $\{\mathbb{Q}_n\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \overline{S_{\mathbb{Q}_n}} = \inf_{\mathbb{P}} \overline{S_{\mathbb{P}}} = \overline{\int_a^b} f(x) dx,$$

and similarly there is a sequence $\{\mathbb{Q}_n'\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \underline{S_{\mathbb{Q}_n'}} = \inf_{\mathbb{P}} \underline{S_{\mathbb{P}}} = \underline{\int_a^b} f(x) dx.$$

Now let

$$\mathbb{P}_n = (\mathbb{Q}_1 \cup \mathbb{Q}_2 \cup \cdots \cup \mathbb{Q}_n) \cup (\mathbb{Q}_1' \cup \cdots \cup \mathbb{Q}_n').$$

Then $\mathbb{P}_n \subset \mathbb{P}_{n+1}$ and

$$\lim_{n \to \infty} \int_{[a,b]} \overline{g_{\mathbb{P}_n}} dm = \lim_{n \to \infty} \overline{S_{\mathbb{P}_n}} = \overline{\int_a^b} f(x) dx,$$

since $\overline{\int_a^b} f(x) dx \leq \overline{S_{\mathbb{P}_n}} \leq \overline{S_{\mathbb{Q}_n}} \to \overline{\int_a^b} f(x) dx$. A similar argument shows that

$$\lim_{n \to \infty} \int_{[a,b]} \underline{g_{\mathbb{P}_n}} dm = \underline{\int_a^b} f(x) dx.$$

Now $\{g_{\mathbb{P}_n}\}_{n\in\mathbb{N}}$ is an increasing sequence of functions bounded above by f. so they converge pointwise to some limit $\underline{g}$ and $\underline{g} \leq f$. Similarly $\overline{g_{\mathbb{P}_n}} \to \overline{g}$ pointwise with $\overline{g} \geq f$. Notice that $\underline{g}$ is bounded and hence integrable on $[a, b]$. Thus by the Dominated Convergence Theorem we have

$$\int_{[a,b]} \underline{g} dm = \lim_{n\to\infty} \int_{[a,b]} \underline{g_{\mathbb{P}_n}} dm = \underline{\int_a^b} f(x)dx,$$

and similarly

$$\int_{[a,b]} \overline{g} dm = \overline{\int_a^b} f(x)dx.$$

Since f is Riemann integrable we have

$$\int_{[a,b]} (\overline{g} - \underline{g})dm = 0,$$

and since $\overline{g} - \underline{g} \geq 0$ we have $\overline{g} = \underline{g}$ a.e.. Moreover, $\overline{g} \geq f \geq \underline{g}$ so $f = \overline{g} = \underline{g}$ a.e. and hence f is $\mathcal{L}$-measurable because $(\mathbb{R}, \mathcal{L}, m)$ is complete, and so finally

$$\int_{[a,b]} f dm = \int_{[a,b]} \overline{g} dm = \overline{\int_a^b} f(x)dx = R\!\!\!\int_a^b f(x)dx.$$

As a sketch of (ii) define the functions

$$H(x) = \lim_{\delta\to 0} \sup_{|x-y|<\delta} f(y) \qquad \text{and} \qquad h(x) = \lim_{\delta\to 0} \inf_{|x-y|<\delta} f(y).$$

It is a standard exercise in Analysis to show that f is continuous if and only if $H(x) = h(x)$. Furthermore one can show that $H = \overline{g}$ and $h = \underline{g}$. With these two facts we have

$$\int_{[a,b]} H dm = \overline{\int_a^b} f(x)dx \qquad \text{and} \qquad \int_{[a,b]} h dm = \underline{\int_a^b} f(x)dx,$$

hence f is Riemann integrable if and only if $H = h$ a.e. if and only if f is continuous a.e.. $\square$

2.5 Complex valued functions

Definition 2.26. Let $(X, \mathcal{M}, \mu)$ be a measure space. A function $f : X \to \mathbb{C}$ is **measurable** if $\mathrm{Re}(f), \mathrm{Im}(f) : X \to \mathbb{R}$ are measurable. A measurable $f : X \to \mathbb{C}$ is **integrable** on $E \in \mathcal{M}$ if $|f| \in L^1(E, X, \mathcal{M}, \mu)$, namely $\int_E |f| d\mu < \infty$. We define $\int_E f d\mu = \int_E \mathrm{Re}(f) d\mu + i \int_E \mathrm{Im}(f) d\mu$.

Note 2.27. Since $|f| \leq |\mathrm{Re}(f)| + |\mathrm{Im}(f)| \leq 2|f|$, we get that f is integrable if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are integrable in the real sense.

Note 2.28. The space $L^1(X; \mathbb{C})$ is a complex-valued vector space and the map from $L^1 \to \mathbb{C}$ given by $f \mapsto \int_X f d\mu$ is linear. Moreover

- $\int_X |f| d\mu \geq 0$ for all $f \in L^1$
- $\int_X |\lambda f| d\mu = |\lambda| \int_X |f| d\mu$

- $\int_X |f+g|d\mu \leq \int_X |f|+|g|d\mu = \int_X |f|d\mu + \int_X |g|d\mu$

However, $\int |f|d\mu = 0$ if and only if $f=0$ **a.e.**, so $f \mapsto \int_X |f|d\mu$ almost satisfies the requirements to be a norm, but not quite. Because of this we'll redefine L^1 as

$$L^1(X;\mathbb{C}) = \{\text{equivalence classes of almost-everywhere defined integrable functions on } X\},$$

where $f \sim g$ if $f=g$ a.e. on X.

Fact: $L^1(X;\mathbb{C})$ is a complete (in the sense that Cauchy sequences converge) normed vector space.

2.6 Modes of Convergence

Definition 2.29. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of complex valued functions on X. Then

- $f_n \to f$ pointwise if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$
- $f_n \to f$ uniformly if $\lim_{n\to\infty} \sup\{|f_n(x)-f(x)| : x \in X\} = 0$

If $(X,\mathcal{M},\mu)$ is a measure space

- $f_n \to f$ a.e. on X if $\mu\left(\{x\in X : \lim_{n\to\infty} f_n(x) \neq f(x)\}\right) = 0$
- $f_n \to f$ in L^1 if $\lim_{n\to\infty} \int_X |f_n - f|d\mu = 0$
- $f_n \to f$ in measure if $\lim_{n\to\infty} \mu\left(\{x \in X : |f_n(x)-f(x)| \geq \varepsilon\}\right) = 0$ for all $\varepsilon > 0$

Also for $p \geq 1$ we say $f_n \to f$ in L^p if $\lim_{n\to\infty} \int_X |f_n - f|^p d\mu = 0$

Note 2.30. We already know that

$$\text{uniform} \implies \text{point-wise} \implies \text{almost everywhere.}$$

Moreover the converse implications are false. For example $\chi_{(0,1/n)} \to 0$ point-wise but not uniformly, and $\chi_{[0,1/n)} \to 0$ a.e. but not point-wise.

Proposition 2.31. If $f_n \to f$ in L^1, then $f_n \to f$ in measure. But the converse is false.

Proof. Let $\varepsilon > 0$, and let $E_{n,\varepsilon} = \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$. Then

$$\int_X |f_n - f|d\mu \geq \int_{E_{n,\varepsilon}} |f_n - f|d\mu \geq \varepsilon\mu(E_{n,\varepsilon}).$$

Therefore

$$\mu(E_{n,\varepsilon}) \leq \frac{1}{\varepsilon}\int_X |f_n - f|d\mu \xrightarrow{n\to\infty} 0,$$

and so $f_n \to f$ in measure.

For a counter example to the converse, notice that $n\chi_{(0,1/n)} \to 0$ in measure but not in L^1. $\square$

Proposition 2.32. If $f_n \to f$ uniformly, then $f_n \to f$ in measure. But the converse is false.

Proof. Let $\varepsilon > 0$. By uniform convergence there is $N \in \mathbb{N}$ such that for $n \geq N$:

$$\{x \in X : |f_n(x) - f(x) \geq \varepsilon\} = \emptyset \qquad \text{in particular } \mu\left(\{x \in X : |f_n(x) - f(x) \geq \varepsilon\}\right) = 0.$$

For a counter example to the converse, notice that $\chi_{(0,1/n)} \to 0$ in measure but not uniformly. □

Proposition 2.33.

(i) If $f_n \to f$ uniformly and $\mu(X) < \infty$, then $f_n \to f$ in L^1

(ii) If $f_n \to f$ a.e. on X and $|f_n| \leq g$ with $g \in L^1$, then $f_n \to f$ in L^1

Proof.

(i) $\displaystyle\int_X |f_n - f| d\mu \leq \mu(X) \underbrace{\sup\{|f_n(x) - f(x)| : x \in X\}}_{\to 0 \text{ by uniform conv.}} \to 0$

(ii) For all $n \in \mathbb{N}$, let $h_n = f_n - f$. Then $h_n \to 0$ a.e. and $|h_n| \leq 2g$ for all $n \in \mathbb{N}$ with $2g \in L^1$. So by the D.C.T. we have

$$\lim_{n\to\infty} \int_X |h_n| d\mu = 0.$$

□

Note 2.34. In (i) $\mu(X) < \infty$ is needed since for example $1/n\chi_{(0,n)} \to 0$ uniformly but not in L^1. Also note the reverse of (ii) does not hold. As a counter-example take $f_n = \chi_{[j/2^n,(j+1)/2^n]}$ for $n = 2^k + j$, $j = 0, \ldots, 2^{k-1}$ on $[0, 1]$. Then $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$ and so on. Then for any $x \in [0, 1]$ we have

$$|\{n \in \mathbb{N} : f_n(x) = 1\}| = \infty \qquad \text{and} \qquad |\{n \in \mathbb{N} : f_n(x) = 0\}| = \infty.$$

So $f_n(x)$ does not converge. However $f_n \to 0$ in L^1.

Theorem 2.35.

(i) If $f_n \to f$ in measure, then there is a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ a.e.

(ii) If $f_n \to f$ in L^1, then there is a subsequence $f_{n_k} \to f$ a.e.

Proof. Let $\varepsilon > 0$, then by convergence in measure we have for all $n \in \mathbb{N}$, there is $k_n \in \mathbb{N}$ such that

$$\mu\left(\{x \in X : |f_k(x) - f(x)| \geq 1/2^n\}\right) < \frac{1}{2^n} \qquad \text{for all } k \geq k_n.$$

Let

$$A_n = \{x \in X : |f_{k_n}(x) - f(x)| \geq 1/2^n\} \qquad \text{and} \qquad E_m = \bigcup_{n\geq m} A_n.$$

Then $\mu(E_m) \leq \sum_{n\geq m} \mu(A_n) \leq \frac{1}{2^{m-1}}$. Let $x \in X \setminus E_m$. Then $x \notin A_n$ for any $n \geq m$, namely $|f_{k_n}(x) - f(x)| < 1/2^n$ for all $n \geq m$. Hence for all $m \in \mathbb{N}$, the subsequence $\{f_{k_n}\}_{n\in\mathbb{N}}$ converges pointwise to f. Now $\mu(E_1) < 1 < \infty$ and $E_1 \supset E_2 \supset \cdots$, so

$$\mu\left(\bigcap_{m=1}^{\infty} E_m\right) = \lim_{m\to\infty} \mu(E_m) = 0.$$

In particular $E = \bigcap_{m=1}^{\infty} E_m$ is a null set and f_{k_n} converges pointwise on $\bigcup_m (X \setminus E_m) = X \setminus E$. Moreover, (ii) follows from Proposition 2.31. □

Theorem 2.36. *(Egorov)* Let $(X, \mathcal{M}, \mu)$ with $\mu(X) < \infty$ and let $f_n, f : X \to \mathbb{R}$ be measurable with $f_n \to f$ a.e. Then for any $\varepsilon > 0$ there is $E \in \mathcal{M}$ with $\mu(E) < \varepsilon$ such that $f_n \to f$ uniformly on $X \setminus E$. We say that $f_n \to f$ **almost uniformly**.

Proof. Let $\varepsilon > 0$. For any $n, m \in \mathbb{N}$, let

$$E_{n,m} = \bigcup_{j=m}^{\infty} \left\{ x \in X : |f_j(x) - f(x)| \geq \frac{1}{2^n} \right\}.$$

Then $E_{n,m} \in \mathcal{M}$, $E_{n,m} \supset E_{n,m+1}$, and $\mu(E_{n,1}) < \mu(X) < \infty$. Hence

$$\lim_{m\to\infty} \mu(E_{n,m}) = \mu\left(\bigcap_{m=1}^{\infty} E_{n,m} \right).$$

If $x \in \bigcap_m E_{n,m}$, then $\{f_j(x)\}_{j\in\mathbb{N}}$ does not converge to $f(x)$, so by a.e. convergence we have

$$\mu\left(\bigcap_m E_{n,m} \right) = 0 \qquad \text{for all } n \in \mathbb{N}.$$

In particular, for any $n \in \mathbb{N}$ there exists $N(n)$ such that $\mu(E_{n,m}) < \varepsilon/2^n$ for all $m \geq N(n)$. Defining $E = \bigcup_{n=1}^{\infty} E_{n,N(n)} \in \mathcal{M}$ we have

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_{n,N(n)}) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Now

$$E^c = \bigcap_{n=1}^{\infty} \bigcap_{j\geq N(n)} \left\{ x \in X : |f_j(x) - f(x)| < \frac{1}{2^n} \right\},$$

so $x \in E^c$ implies that for all $n \in \mathbb{N}$ and all $j \geq N(n)$ we have

$$|f_j(x) - f(x)| < \frac{1}{2^n}.$$

Namely $f_j \to f$ uniformly on E^c with $\mu(E) < \infty$. □

Note 2.37. The assumption $\mu(X) < \infty$ is necessary. Consider $\chi_{[n,n+1]} \to 0$ pointwise in $\mathbb{R}$, but if $\chi_{[n,n+1]}$ uniformly on E^c, then E^c must be bounded above and so $\mu(E) = \infty$.

Proposition 2.38. Let $\mu(X) < \infty$. If $f_n \to f$ a.e., then $f_n \to f$ in measure.

Proof. By Egorov's Theorem, for all $\varepsilon > 0$, there is $E \in \mathcal{M}$ with $\mu(E) < \varepsilon$ and such that there is $N \in \mathbb{N}$ with

$$\sup \{|f_n(x) - f(x)| : x \in E^c\} < \varepsilon \qquad \text{for all } n \geq N.$$

Equivalently,

$$\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\} \subset E,$$

which yields the claim upon taking $\mu(\cdot)$ of both sides. □

3 Product measures

3.1 Product measures

Definition 3.1. Let $(X, \mathcal{M}, \mu)$, and $(Y, \mathcal{N}, \nu)$ be measure space. A set of the form $A \times B$ with $A \in \mathcal{M}$, $B \in \mathcal{N}$ is called a **rectangle**. Let

$$\mathcal{R} = \left\{ \bigcup_{i=1}^{n} A_i \times B_i : A_i \in \mathcal{M}, B_i \in \mathcal{N}; (A_j \times B_j) \cap (A_k \times B_k) = \emptyset \text{ for } j \neq k, \text{ and } n \in \mathbb{N} \right\}.$$

Note 3.2. Since any finite union can be written as a finite disjoint union, the disjointedness condition can be dropped. Hence $\mathcal{R}$ is closed under finite unions. Moreover

$$(A \times B)^c = (X \times B^c) \cup (A^c \times Y) \in \mathcal{R},$$

so $\mathcal{R}$ is closed under complements. In total we see that $\mathcal{R}$ is an algebra.

Lemma 3.3. π is a premeasure, where π is defined by $\pi : \mathcal{R} \to [0, \infty]$ by

$$\pi\left(\bigcup_{i=1}^{n} (A_i \times B_i)\right) = \sum_{i=1}^{n} \mu(A_i)\nu(B_i),$$

whenever $A_i \in \mathcal{M}$ and $B_i \in \mathcal{N}$ and $(A_j \times B_j) \times (A_k \times B_k) = \emptyset$ for $j \neq k$. Here we let $0 \cdot \infty = 0$.

Proof. Let $\{A_j \times B_j\}_{j \in \mathbb{N}}$ be a countable collection of disjoint rectangles such that

$$\bigcup_{j=1}^{\infty} A_j \times B_j \in \mathcal{R}.$$

By definition of $\mathcal{R}$, there is a finite collection so that

$$\bigcup_{j=1}^{\infty} A_j \times B_j = \bigcup_{i=1}^{n} \tilde{A}_i \times \tilde{B}_i,$$

so by disjointedness:

$$\sum_{i=1}^{n} \chi_{\tilde{A}_i}(x)\chi_{\tilde{B}_i}(y) = \sum_{i=1}^{n} \chi_{\tilde{A}_i \times \tilde{B}_i}(x, y) = \chi_{\bigcup_{i=1}^{n} \tilde{A}_i \times \tilde{B}_i}(x, y) = \chi_{\bigcup_{j=1}^{\infty} A_j \times B_j}(x, y) = \sum_{j=1}^{\infty} \chi_{A_j}(x)\chi_{B_j}(y).$$

Using the Monotone Convergence Theorem (for the RHS) we integrate over X w.r.t. μ to obtain

$$\sum_{i=1}^{n} \mu(\tilde{A}_i)\chi_{\tilde{B}_i}(y) = \sum_{j=1}^{\infty} \mu(A_j)\chi_{B_j}(y).$$

Then integrating over Y w.r.t. ν yields

$$\pi\left(\bigcup_{j=1}^{\infty} A_j \times B_j\right) = \pi\left(\bigcup_{i=1}^{n} \tilde{A}_i \times \tilde{B}_i\right) = \sum_{i=1}^{n} \mu(\tilde{A}_i)\nu(\tilde{B}_i) = \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j) = \sum_{j=1}^{\infty} \mu(A_j \times B_j).$$

Finally $\pi(\emptyset) = \mu(\emptyset \times \emptyset) = \mu(\emptyset)\nu(\emptyset) = 0$. $\square$

Definition 3.4. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. We define the **product σ-algebra**, denoted $\mathcal{M} \otimes \mathcal{N}$, as the σ-algebra generated by $\mathcal{R}$. And the **product measure** is $\mu \times \nu := \pi^\star \restriction_{\mathcal{M}\otimes\mathcal{N}}$.

Lemma 3.5. $\mu \times \nu$ is σ-finite if μ and ν are σ-finite.

Proof. By σ-finiteness, write $\{A_j\}$ in $\mathcal{M}$ with $\mu(A_j) < \infty$ and $X = \bigcup_j A_j$. Similarly $\{B_j\}$ in $\mathcal{N}$ with $\nu(B_j) < \infty$ and $Y = \bigcup_j B_j$. Then $A_j \times B_k \in \mathcal{R} \subset \mathcal{M} \otimes \mathcal{N}$ and $X \times Y = \bigcup_{j,k} A_j \times B_k$. Moreover $(\mu \times \nu)(A_j \times B_k) = \mu(A_j)\nu(B_k) < \infty$. $\square$

Proposition 3.6. Let (X, d_X) an (Y, d_Y) be separable metric spaces. Define

$$D((x,y),(x',y')) = \left(d_X(x,x')^2 + d_Y(y,y')^2\right)^{1/2}.$$

Then $(X \times Y, D)$ is a metric space and $\mathcal{B}(X \times Y) = \mathcal{B}(X) \otimes \mathcal{B}(Y)$. [A metric space is **separable** if it has a countable dense set].

Proof. Let $\mathcal{O}$ be the open sets in $X \times Y$ and recall

$$\mathcal{R} = \left\{ \bigcup_{i=1}^n A_i \times B_i : A_i \in \mathcal{M}, B_i \in \mathcal{N}; (A_j \times B_j) \cap (A_k \times B_k) = \emptyset \text{ for } j \neq k, \text{ and } n \in \mathbb{N} \right\}.$$

$\mathcal{B}_{X\times Y} = \mathcal{M}(\mathcal{O})$ and $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{M}(\mathcal{R})$. To show $\mathcal{B}_{X\times Y} \subset \mathcal{B}_X \otimes \mathcal{B}_Y$ it suffices to show $\mathcal{O} \subset \mathcal{B}_X \otimes \mathcal{B}_Y$ by minimality. Similarly to show $\mathcal{B}_X \otimes \mathcal{B}_Y \subset \mathcal{B}_{X\times Y}$ it suffices to show that $\mathcal{R} \subset \mathcal{B}_{X\times Y}$.

- $\mathcal{B}_{X\times Y} \subset \mathcal{B}_X \otimes \mathcal{B}_Y$. Since X and Y are separable there exists countable dense sets, call them S_X and S_Y respectively. Let

  $$\mathcal{C} = \{B_q(s) \times B_p(t) : s \in S_X; t \in S_Y; p, q \in \mathbb{Q} \text{ and } p, q > 0\}.$$

  This collection $\mathcal{C}$ is countable since S_X, S_Y, and $\mathbb{Q}$ are countable, moreover $\mathcal{C} \subset \mathcal{R} \subset \mathcal{B}_X \otimes \mathcal{B}_Y$. Let $U \in \mathcal{O}$ and define $V = \bigcup_{\substack{R \in \mathcal{C} \\ R \subset U}} R$. As a countable union of elements of $\mathcal{C} \subset \mathcal{B}_X \otimes \mathcal{B}_Y$, we have $V \in \mathcal{B}_X \otimes \mathcal{B}_Y$. I claim that $U = V$. It's clear that $V \subset U$. For the other inclusion, let $(x, y) \in U$. Since U is open there exists $r > 0$ such that

  $$B_r((x,y)) = \{(x',y') : \sqrt{d_X(x,x')^2 + d_Y(y,y')^2} < r\} \subset U.$$

  Moreover, $B_{r/2}(x) \times B_{r/2}(y) \subset B_r((x,y)) \subset U$. Since S_X is dense in X, there is a point $s \in S_X$ with $d_X(s,x) < r/4$, and since the rationals are dense in the reals, there is $q \in \mathbb{Q}$ such that $d_X(s,x) < q < r/4$. Then the ball $B_q(s) \ni x$ and $B_q(s) \subset B_{r/2}(x)$. The first point is clear because $d_X(s,x) < q$. For the second point, let $\tilde{x} \in B_q(s)$, hence $d_X(\tilde{x}, s) < q$. Now $d_X(\tilde{x}, x) \leq d_X(\tilde{x}, s) + d_X(s, x) < r/4 + r/4 = r/2$, so $\tilde{x} \in B_{r/2}(x)$ and $B_q(s) \subset B_{r/2}(x)$. Similarly there is $t \in S_Y$ and $p \in \mathbb{Q}$ such that $y \in B_p(t) \subset B_{r/2}(y)$. Hence $(x, y) \in B_q(s) \times B_p(t) \subset B_{r/2}(x) \times B_{r/2}(y) \subset B_r((x,y)) \subset U$. And since $B_q(s) \times B_p(t) \subset V$ we conclude that $(x, y) \in V$ and moreover $U \subset V$. In total we have shown that any open set in $\mathcal{B}_{X\times Y}$ is a countable union of sets in $\mathcal{C} \subset \mathcal{B}_X \otimes \mathcal{B}_Y$. And so $\mathcal{O} \subset \mathcal{B}_X \otimes \mathcal{B}_Y$ as desired.

- $\mathcal{B}_X \otimes \mathcal{B}_Y \subset \mathcal{B}_{X\times Y}$. Consider the functions $f : X \times Y \to X$ given by $f(x,y) = x$ and $g : X \times Y \to Y$ given by $g(x,y) = y$. We claim that f and g are $(\mathcal{B}_{X\times Y}, \mathcal{B}_\cdot)$-measurable, for which it suffices to show that they are continuous. We show that f is continuous. Let $U \subset X$ be open, then $f^{-1}(U) = U \times Y$ which is also open. Indeed, let $(x,y) \in U \times Y$. Since U is open, there exists $r > 0$ such that

  $$B_r(x) = \{x' \in X : d_X(x,x') < r\} \subset U.$$

  Let $(x',y') \in B_r((x,y))$. We claim $B_r((x,y)) \subset U \times Y$. To this end let $(x',y') \in B_r((x,y))$. Therefore

  $$D((x,y),(x',y')) = \sqrt{d_X(x,x') + d_Y(y,y')} < r,$$

  which implies

  $$d_X(x,x')^2 \leq d_X(x,x')^2 + d_Y(y,y')^2 < r^2.$$

  In particular $x' \in B_r(x) \subset U$. Therefore we see that $B_r((x,y)) \subset U \times Y$. Namely $U \times Y$ is open. In total we see that f is continuous and hence $(\mathcal{B}_{X\times Y}, \mathcal{B}_X)$-measurable. The same argument works for g. Now let $A \in \mathcal{B}_X$ and $B \in \mathcal{B}_Y$. Then by measurability

  $$A \times B = (A \times Y) \cap (X \times B) = f^{-1}(A) \cap g^{-1}(B) \in \mathcal{B}_{X\times Y}.$$

  Hence finite unions of the form $\bigcup_{i=1}^n A_i \times B_i \in \mathcal{B}_{X\times Y}$ also. In particular $\mathcal{R} \subset \mathcal{B}_{X\times Y}$.

$\square$

Proposition 3.7. Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be measurable spaces. Further let $x \in X$, $y \in Y$, and $E \in \mathcal{M} \otimes \mathcal{N}$. Then

(i) $E_x = \{y' \in Y : (x,y') \in E\} \in \mathcal{N}$ and $E^y = \{x' \in X : (x',y) \in E\} \in \mathcal{M}$.

(ii) If $f : X \times Y \to \mathbb{R}$ is $\mathcal{M} \otimes \mathcal{N}$ measurable, then $f_x : Y \to \mathbb{R}$ given by $f_x(y) = f(x,y)$ is $\mathcal{N}$ measurable. And similarly $f^y : X \to \mathbb{R}$ given by $f^y(x) = f(x,y)$ is $\mathcal{M}$ measurable.

Proof.

(i) Let $\mathcal{P} = \{E \subset \mathcal{M} \otimes \mathcal{N} : E_x \in \mathcal{N}, E^y \in \mathcal{M} \text{ for all } x \in X, y \in Y\}$. Notice that for any measurable rectangle $(A \times B)_x = \begin{cases} B & x \in A \\ \emptyset & x \notin A \end{cases}$ and similarly for y. So $A \times B \in \mathcal{P}$, namely the set of measurable rectangles $\mathcal{R} \subset \mathcal{P}$. Moreover $\mathcal{P}$ is σ-algebra. Indeed it is closed under complement since for $E \in \mathcal{P}$, $(E^c)_x = (E_x)^c \in \mathcal{N}$ and similarly for y. It is closed under countable union since for $\{E_j\}_{j\in\mathbb{N}}$ in $\mathcal{P}$ we have $(\bigcup_j E_j)_x = \bigcup_j (E_j)_x \in \mathcal{N}$ (similarly for y). So $\mathcal{M} \otimes \mathcal{N} \subset \mathcal{P}$ since $\mathcal{M} \otimes \mathcal{N}$ is the smallest σ-algebra containing $\mathcal{R}$.

(ii) Let $B \in \mathcal{B}(\mathbb{R})$. Since f is $\mathcal{M} \otimes \mathcal{N}$ measurable, $f^{-1}(B) \in \mathcal{M} \otimes \mathcal{N}$. But then by (i) we have

$$f_x^{-1}(B) = \{y \in Y : f_x(y) = f(x,y) \in B\} = (f^{-1}(B))_x \in \mathcal{N}.$$

$\square$

3.2 Monotone Classes

Definition 3.8. Let X be a non-empty set. A collection $\mathcal{C} \subset \mathcal{P}(X)$ is a **monotone class** if both

(i) $\mathcal{C}$ is closed under countable increasing unions, namely for $\{E_j\}_{j\in\mathbb{N}}$ in $\mathcal{C}$ with $E_j \subset E_{j+1}$ we have $\bigcup_{j=1}^{\infty} E_j \in \mathcal{C}$

(ii) and $\mathcal{C}$ is closed under countable decreasing intersections, namely for $\{E_j\}_{j\in\mathbb{N}}$ in $\mathcal{C}$ with $E_j \supset E_{j+1}$ we have $\bigcap_{j=1}^{\infty} E_j \in \mathcal{C}$.

Note 3.9. A σ-algebra is a monotone class. Moreover if I is any index set, and $\{\mathcal{C}_i, i \in I\}$ are monotone classes, then $\bigcap_{i\in I} \mathcal{C}_i$ is a monotone class. In particular for any $\mathcal{E} \subset \mathcal{P}(X)$,

$$\mathcal{C}(\mathcal{E}) = \bigcap_{\substack{\mathcal{C} \text{ mon. class} \\ \mathcal{E}\subset\mathcal{C}}} \mathcal{C}$$

is the smallest monotone class containing $\mathcal{E}$ and is called the monotone class **generated** by $\mathcal{E}$.

Lemma 3.10. *(Monotone Class Lemma)* If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, then $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$

Proof. $\mathcal{C}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$ since $\mathcal{M}(\mathcal{A})$ is a σ-algebra and hence a monotone class. It also contains $\mathcal{A}$ and so conclude by minimality. For the other inclusion we claim that $\mathcal{C}(\mathcal{A})$ is a σ-algebra, from which the claim follows by minimality.

Any algebra closed under countable increasing unions is a σ-algebra (for $\{F_j\}_{j\in\mathbb{N}}$ in $\mathcal{A}$ consider $E_n = \bigcup_{j=1}^{n} F_j \in \mathcal{A}$ which is increasing and $\bigcup_j F_j = \bigcup_n E_n$). So it suffices to show that for any $E, F \in \mathcal{C}(\mathcal{A})$ we have $E \setminus F$, $F \setminus E$, $E \cap A \in \mathcal{C}(\mathcal{A})$ from which it follows that $\mathcal{C}(\mathcal{A})$ is an algebra (since $X \in \mathcal{A}$ and $\mathcal{A} \subset \mathcal{C}(\mathcal{A})$ so $F^c = X \setminus F$ and $E \cup F = (E^c \cap F^c)^c \in \mathcal{C}(\mathcal{A})$).

For $E \in \mathcal{C}(\mathcal{A})$, let

$$\mathcal{D}(E) = \{F \in \mathcal{C}(\mathcal{A}) : E \setminus F, F \setminus E, F \cap E \in \mathcal{C}(\mathcal{A})\}.$$

With that it suffices to show that if $E \in \mathcal{C}(\mathcal{A})$, then $\mathcal{C}(\mathcal{A}) \subset \mathcal{D}(E)$. For this it suffices to show that $\mathcal{D}(E)$ is a monotone class containing $\mathcal{A}$.

(i) $E \in \mathcal{C}(\mathcal{A}) \implies \emptyset, E \in \mathcal{D}(E)$ since $\emptyset \in \mathcal{A} \subset \mathcal{C}(\mathcal{A})$, in particular $\mathcal{D}(E) \neq \emptyset$.

(ii) For $E, F \in \mathcal{C}(\mathcal{A})$ we have $F \in \mathcal{D}(E) \iff E \in \mathcal{D}(F)$ by symmetry.

(iii) $\mathcal{D}(E)$ is closed under countable increasing unions: Let $\{F_n\}_{n\in\mathbb{N}}$ be in $\mathcal{D}(E)$ with $F_n \subset F_{n+1}$ and $F = \bigcup_{n=1}^{\infty} F_n$. Then

- $E \setminus F_n = E \cap F_n^c \in \mathcal{C}(\mathcal{A})$ by definition of $\mathcal{D}(E)$ and is decreasing
- $F_n \setminus E = F_n \cap E^c \in \mathcal{C}(\mathcal{A})$ by definition of $\mathcal{D}(E)$ and is increasing
- $F_n \cap E \in \mathcal{C}(\mathcal{A})$ by definition of $\mathcal{D}(E)$ and is increasing

Hence

- $E \setminus F = E \cap (\bigcap_n F_n^c) = \bigcap_n (E \cap F_n^c) \in \mathcal{C}(\mathcal{A})$ since $\mathcal{C}(\mathcal{A})$ is a monotone class
- $F \setminus E = (\bigcup_n F_n) \cap E^c = \bigcup_n (F_n \cap E^c) \in \mathcal{C}(\mathcal{A})$ since $\mathcal{C}(\mathcal{A})$ is a monotone class
- $F \cap E = (\bigcup_n F_n) \cap E = \bigcup_n (F_n \cap E) \in \mathcal{C}(\mathcal{A})$ since $\mathcal{C}(\mathcal{A})$ is a monotone class

So $F \in \mathcal{D}(E)$, that is, $\mathcal{D}(E)$ is closed under countable increasing unions.

(iv) $\mathcal{D}(E)$ is closed under countable decreasing intersection by a similar argument

So $\mathcal{D}(E)$ is a monotone class by (iii) and (iv). Moreover, let $A \in \mathcal{A}$ then $\mathcal{A} \subset \mathcal{D}(A)$ since $\mathcal{A}$ is an algebra so for any $F \in \mathcal{A}$ we have $A \setminus F$,$F \setminus A$, and $F \cap A$ are all in $\mathcal{A} \subset \mathcal{C}(\mathcal{A})$ and so $F \in \mathcal{D}(A)$. Moreover, $\mathcal{C}(\mathcal{A}) \subset \mathcal{D}(A)$ since $\mathcal{D}(A)$ is a monotone class. Furthermore: $E \in \mathcal{C}(A)$ so $E \in \mathcal{D}(A)$, and then by (ii) we have $A \in \mathcal{D}(E)$. That is, $\mathcal{A} \subset \mathcal{D}(E)$ as desired. □

3.3 The Fubini-Tonelli Theorems

Proposition 3.11. Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let $E \in \mathcal{M} \otimes \mathcal{N}$. Then the functions $f : X \to [0, \infty]$ given by $f(x) = \nu(E_x)$ and $g : Y \to [0, \infty]$ given by $g(y) = \mu(E^y)$. are $\mathcal{M}$, respectively $\mathcal{N}$, measurable and

$$(\mu \times \nu)(E) = \int_X f d\mu = \int_Y g d\mu.$$

Proof. Let $\mathcal{C} = \{E \in \mathcal{M} \otimes \mathcal{N} : \text{ the proposition holds}\}$. We claim that $\mathcal{C}$ is a monotone class containing $\mathcal{R}$ in which case:

$$\mathcal{M} \otimes \mathcal{N} = \mathcal{M}(\mathcal{R}) = \mathcal{C}(\mathcal{R}) \subset \mathcal{C}.$$

We prove the claim first when μ and ν are finite measures. Let $A \in \mathcal{M}$, $B \in \mathcal{N}$, and $E = A \times B \in \mathcal{R}$. Then $E \in \mathcal{C}$ since

$$\nu((A \times B)_x) = \nu\left(\left\{\begin{smallmatrix} B & x \in A \\ \emptyset & x \notin A \end{smallmatrix}\right\}\right) \chi_A \nu(B)$$

which is $\mathcal{M}$-measurable and

$$\int_X \nu((A \times B)_x) d\mu(x) = \nu(B) \int_X \chi_A d\mu = \nu(B)\mu(A) = (\mu \times \nu)(A \times B).$$

Moreover $\mathcal{C}$ is closed under finite disjoint unions. If $E, F \in \mathcal{C}$ disjoint, then $(E \cup F)_x = E_x \cup F_x$, hence $\nu(E_x \cup F_x) = \nu(E_x) + \nu(F_x)$ and we can conclude by linearity of the integral. Thus $\mathcal{R} \subset \mathcal{C}$.

We now show that $\mathcal{C}$ is a monotone class.

- $\mathcal{C}$ is closed under countable increasing unions. Let $E_1 \subset E_2 \subset \cdots$ be in $\mathcal{C}$ and let $E = \bigcup_j E_j$. Then $f_n(x) := \nu((E_n)_x)$ is an increasing sequence converging pointwise to $f(x) := \nu(E_x)$ by continuity from below of ν. Hence f is measurable and by the M.C.T.

$$\int_X f d\mu = \lim_{n \to \infty} \int_X f_n d\mu = \lim_{n \to \infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E),$$

  with the final equality following from continuity of $\mu \times \nu$.

- $\mathcal{C}$ is closed under countable decreasing intersections. Let $E_1 \supset E_2 \supset \cdots$ be in $\mathcal{C}$ and let $E = \bigcap_j E_j$. Then $f_n(x) := \nu((E_n)_x)$ is a decreasing sequence of functions converging pointwise to $f(x) := \nu(E_x)$ by continuity from above (here we needed $\nu((E_1)_x) < \nu(Y) < \infty$). Hence f is measurable and $0 \le f_n(x) \le f_1(x) \in L^1(X)$ so by the D.C.T. and continuity of $\mu \times \nu$

$$\int_X f d\mu = \lim_{n \to \infty} f_n d\mu = \lim_{n \to \infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E).$$

Now if μ and ν are σ-finite, then $X \times Y$ can be written as a countable increasing union of rectangles $\{X_n \times Y_n : n \in \mathbb{N}\}$ of finite measure. Let $E \in \mathcal{M} \otimes \mathcal{N}$. We apply the finite case to $E \cap (X_n \times Y_n)$ to obtain

$$(\mu \times \nu)(E \cap (X_n \times Y_n)) = \int_X \nu((E \cap (X_n \times Y_n))_x) d\mu(x) = \int_X \nu(E_x \cap Y_n)\chi_{X_n}(x) d\mu(x).$$

Letting $n \to \infty$, the LHS converges to $(\mu \times \nu)(E)$ by continuity from below and the RHS converges to $\int_X \nu(E_x) d\mu(x)$ by M.C.T. $\square$

Theorem 3.12. *(Tonelli)* Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let the function $f : X \times Y \to [0, \infty]$ be $\mathcal{M} \otimes \mathcal{N}$-measurable. Then $g : X \to [0, \infty]$ given by $g(x) = \int f(x, y) d\nu(y)$ is $\mathcal{M}$-measurable, $h : Y \to [0, \infty]$ given by $h(y) = \int f(x, y) d\mu(x)$ is $\mathcal{N}$-measurable, and

$$\int f d(\mu \times \nu) = \int g d\mu = \int h d\nu.$$

Proof. If f is a non-negative simple function, apply Proposition 3.11 and linearity of the integral. In the general case $f \in L^+(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$, let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of non-negative simple functions increasing pointwise to f. For example take

$$f_n = \sum_{m=0}^{4^n} \frac{m}{2^n} \chi_{f^{-1}([m/2^n, (m+1)/2^n])} + 2^n \chi_{f^{-1}([2^n, \infty))}.$$

Then by the M.C.T., the limit of $\int f_n d(\mu \times \nu) = \int (\int f_n d\nu) d\mu$ is $\int f d(\mu \times \nu) = \int (\int f d\nu) d\mu$. $\square$

Corollary 3.13. *(Fubini)* Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let the function $f : X \times Y \to \mathbb{R}$ be $\mathcal{M} \otimes \mathcal{N}$-measurable. Suppose further that $f \in L^1(\mu \times \nu)$, then

- $f_x : Y \to \mathbb{R}$ given by $f_x(y) = f(x, y)$ is in $L^1(\nu)$ for almost all $x \in X$
- $g : X \to \mathbb{R}$ given by $g(x) = \int f_x d\nu$ is in $L^1(\mu)$
- $f^y : X \to \mathbb{R}$ given by $f^y(x) = f(x, y)$ is in $L^1(\mu)$ for almost all $y \in Y$
- $h : Y \to \mathbb{R}$ given by $h(y) = \int f^y d\mu$ is in $L^1(\nu)$
- and
$$\int f d(\mu \times \nu) = \int g d\mu = \int h d\nu.$$

Note 3.14. Fubini also holds for complex-valued functions.

Proof. Since $f \in L^1(\mu \times \nu)$ we have $\int |f| d(\mu \times \nu) < \infty$, so by Tonelli we have

$$\int \left(\int |f(x, y)| d\mu(x) \right) d\nu(y) < \infty.$$

Namely $f^y \in L^1(\mu)$ for almost all $y \in Y$ and $|h(y)| \leq \int |f(x, y)| d\mu(x) \in L^1(\nu)$. Finally, the equality of the integrals follows from Tonelli applied to both $\max\{f, 0\}$ and $-\max\{-f, 0\}$. $\square$

4 Differentiation

4.1 Signed measures

Definition 4.1. Let $(X, \mathcal{M})$ be a measurable space.

(i) A **signed measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \overline{\mathbb{R}}$ such that

- $\nu(\emptyset) = 0$
- ν assumes only one the values $\pm\infty$
- If $\{E_n\}_{n\in\mathbb{N}}$ are disjoint with $E_n \in \mathcal{M}$, then

$$\nu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \nu(E_n)$$

where the series must be absolutely convergent if the LHS is finite.

(ii) A set $E \in \mathcal{M}$ is **positive**, resp. **negative/null** for ν if for every subset $F \subset E$ with $F \in \mathcal{M}$ we have $\nu(F) \geq 0$, resp. $\nu(F) \leq 0/\nu(F) = 0$.

Example 4.2.

(i) A measure μ on $(X, \mathcal{M})$ is a signed measure and X is positive.

(ii) If μ, ν are finite measures on $(X, \mathcal{M}$, then $\mu - \nu$ is a signed measure

(iii) If μ is a measure on $(X, \mathcal{M}$ and $f : X \to \mathbb{R}$ is measurable with at least one of $\int \max\{f, 0\} d\mu$, $\int \max\{-f, 0\} d\mu$ finite, then $\nu(E) = \int_E f d\mu$ is a signed measure.

⋆ f is called **extended integrable** and we write $d\nu = f d\mu$. A set $E \in \mathcal{M}$ is positive, resp. negative/null, w.r.t. ν if $f(x) \geq$, resp. $f(x) \leq 0/f(x) = 0$ μ-a.e. on E.

Lemma 4.3. Let ν be a signed measure on $(X, \mathcal{M})$. Then ν is continuous. Namely

(i) If $E_1 \subset E_2 \subset \cdots$ in $\mathcal{M}$, then $\nu\left(\bigcup_n E_n\right) = \lim_{n\to\infty} \nu(E_n)$

(ii) If $E_1 \supset E_2 \supset \cdots$ in $\mathcal{M}$ with $\nu(E_1) < \infty$, then $\nu\left(\bigcap_n E_n\right) = \lim_{n\to\infty} \nu(E_n)$

Proof. Exercise (same as the proof for measures) □

Lemma 4.4. Let ν be a signed measure on $(X, \mathcal{M})$

(i) If E is positive for ν and $F \subset E$, $F \in \mathcal{M}$ then F is positive

(ii) If $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of positive sets, then $\bigcup_{n=1}^{\infty} E_n$ is positive

Proof. (i) If $G \subset F$, $G \in \mathcal{M}$, then $G \subset E$ also and so $\nu(G) \geq 0$ since E is positive.

(ii) Let $D_n = E_n \setminus \bigcup_{j=1}^{n-1} E_j$. The D_n's are disjoint, positive by (i), and $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} E_n$. So for $F \subset \bigcup_{n=1}^{\infty} E_n$, $F \in \mathcal{M}$ write $F = \bigcup_{n=1}^{\infty}(F \cap D_n)$. Then by σ-additivity

$$\nu(F) = \sum_{n=1}^{\infty} \underbrace{\nu(F \cap D_n)}_{\geq 0} \geq 0 \qquad \text{since } F \cap D_n \subset D_n.$$

□

Lemma 4.5. Let ν be a signed measure. Let $E \subset \mathcal{M}$ be such that $0 < \nu(E) < \infty$. Then there is $A \subset E$, $A \in \mathcal{M}$ such that A is positive and non-null.

Proof. If E is positive, pick $A = E$. Otherwise there is a subset $F \subset E$, $F \in \mathcal{M}$ such that $\nu(F) < 0$. Pick $E_1 \subset E$, $E_1 \in \mathcal{M}$ so that $\nu(E_1) < -1/n_1$ where n_1 is the smallest integer for which E_1 can be found. Then $\nu(E \setminus E_1) = \nu(E) - \nu(E_1) > 0$. If $E \setminus E_1$ is positive, pick $A = E \setminus E_1$. Otherwise continue recursively setting

$$E_k \subset E \setminus \bigcup_{j=1}^{k-1} E_j \qquad E_k \in \mathcal{M},$$

and $\nu(E_k) < -1/n_k$ where $n_k \in \mathbb{N}$ is the smallest possible. Again $\nu\left(E \setminus \bigcup_{j=1}^{k} E_j\right) = \nu(E) - \sum_{j=1}^{k} \nu(E_j) > 0$. Either the recursion stops with $A = E \setminus \bigcup_{j=1}^{k} E_j$ being positive, or we take $A = E \setminus \bigcup_{j=1}^{\infty} E_j$. The claim is that in this latter case A is positive and non-null. By construction the E_j's are disjoint so $0 < \nu(E) = \nu(A) + \sum_{j=1}^{\infty} \nu(E_j)$, hence $\nu(A) > 0$ since the $\nu(E_j)$'s are strictly negative. In particular A is non-null. Moreover, by the definition of signed measure and the fact that $\nu(E) < \infty$, the series must be absolutely convergent. Hence $1/n_j \to 0$. To prove that A is positive it suffices to show that for every $\varepsilon > 0$ there is no set $B \subset A$, $B \in \mathcal{M}$ with $\nu(B) < -\varepsilon$. Let $k \in \mathbb{N}$ be such that $1/(n_k - 1) < \varepsilon$. Note that $A \subset E \setminus \bigcup_{j=1}^{k-1} E_j$. Recall n_k is the smallest integer such that there is $B \subset E \setminus \bigcup_{j=1}^{k-1} E_j$, $B \in \mathcal{M}$ with $\nu(B) < -1/n_k$. So there is no $B \subset A$ such that $\nu(B) < -1/(n_k - 1)$ and thus there is no B such that $\nu(B) < -\varepsilon$. □

Theorem 4.6. *(Hahn decomposition)* Let ν be a signed measure on $(X, \mathcal{M})$. There is a positive $P \in \mathcal{M}$ and negative $N \in \mathcal{M}$ (w.r.t. ν) such that $P \cap N = \emptyset$ and $P \cup N = X$. Moreover for any other such P' and N' we have that the symmetric differences $P \triangle P'$ and $N \triangle N'$ are null.

Proof. Assume w.l.o.g. that ν does not take the value $+\infty$. Let $m = \sup\{\nu(E) : E \text{ is positive}\}$. There is a sequence $\{\tilde{P}_n\}_{n\in\mathbb{N}}$ such that $\nu(\tilde{P}_n) \to m$. Let $P_n = \bigcup_{j=1}^{n} \tilde{P}_j$. Then P_n is an increasing sequence of positive sets and $\nu(P_n) = \nu(\tilde{P}_n) - \nu(P_n \setminus \tilde{P}_n)$, so $\nu(\tilde{P}_n) \leq \nu(P_n) \leq m$ since P_n is positive. Let $P = \bigcup_{n=1}^{\infty} P_n$, then P is positive and $\nu(P) = \lim_{n\to\infty} \nu(P_n) = m$ be the Squeeze Theorem and continuity from below. Moreover $m < \infty$ since the supremum is attained and ν does not take the value $+\infty$. Let $N = X \setminus P$, then N is negative. First of all, assume that $A \subset N$ is positive and non-null. Then $P \cup A$ is positive and $\nu(P \cup A) = \nu(P) + \nu(A) > \nu(P)$ which contradicts the maximality of P so there is no such A. Now if N was non-negative, there is a $B \subset N$, $B \in \mathcal{M}$ such that $\nu(B) > 0$. By Lemma 4.5 there is $A \subset B$, $A \in \mathcal{M}$ such that A is positive, but such an A doesn't exist so N is negative.

Let P' and N' be another Hahn decomposition. Then $P \setminus P' \subset P$ is positive but $P \setminus P' = P \cap P'^c \subset N'$ is negative. So every subset of $P \setminus P'$ is positive and negative and hence null. Can do the same argument for $P' \setminus P$, $N' \setminus N$, and $N \setminus N'$. Thus

$$P \triangle P' = (P \setminus P') \cup (P' \setminus P) = (N' \setminus N) \cup (N \setminus N') = N \triangle N'$$

are both null. □

Definition 4.7. Two signed measures μ, ν on $(X, \mathcal{M})$ are **mutually singular** if there exists $E, F \in \mathcal{M}$ such that $X = E \cup F$, $E \cap F = \emptyset$ and E is null for μ, F is null for ν. As notation we write $\mu \perp \nu$.

Example 4.8. Let $(X, \mathcal{M}) = (\mathbb{R}, \mathcal{L})$. Then $m \perp \delta_0$ since $m(\{0\}) = 0$ and $\delta_0(\mathbb{R} \setminus \{0\}) = 0$.

Theorem 4.9. *(Jordan decomposition)* Let ν be a signed measure on $(X, \mathcal{M})$. There exist unique positive measures ν_+ and ν_- on $(X, \mathcal{M})$ such that $\nu_+ \perp \nu_-$ and $\nu = \nu_+ - \nu_-$.

Proof. Existence follows from the Hahn decomposition $X = P \cup N$ with $P \cap N = \emptyset$. Taking $\nu_+(A) = \nu(A \cap P)$ and $\nu_-(A) = -\nu(A \cap N)$ works. To show uniqueness, let $\nu = \nu'_+ - \nu'_-$ with $P' \cup N' = X$, $P' \cap N' = \emptyset$ and $\nu'_+(N') = 0$, $\nu'_-(P') = 0$. We show that this is Hahn decomposition. Indeed let $A \subset P'$, then $\nu(A) = \nu'_+(A) - \nu'_-(A) = \nu'_+(A) \geq 0$ so P' is positive. Similarly N' is negative. Now let $A \in \mathcal{M}$, then

$$\begin{aligned}
\nu'_+(A) &= \nu'_+(A \cap P') + \underbrace{\nu'_+(A \cap N')}_{=0} \\
&= \nu'_+(A \cap P') - \underbrace{\nu'_-(A \cap P')}_{=0} \\
&= \nu(A \cap P') \\
&= \nu(A \cap P' \cap P) + \underbrace{\nu(A \cap (\overbrace{P' \setminus P}^{\subset P \triangle P'}))}_{=0} \\
&= \nu(A \cap P' \cap P) \\
&= \cdots \\
&= \nu_+(A)
\end{aligned}$$

Hence $\nu'_+ = \nu_+$ and similarly for $\nu'_- = \nu_-$. $\square$

Definition 4.10. Let ν be a signed measure on $(X, \mathcal{M})$, and let $\nu = \nu_+ - \nu_-$ be its Jordan decomposition. The measure $|\nu| = \nu_+ + \nu_-$ is called the **total variation** of ν.

Example 4.11.

- Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ an extended μ-integrable function. Let $\nu(E) = \int_E f d\mu$. Then

$$\nu_\pm(E) = \int_E f_\pm d\mu \qquad \text{and} \qquad |\nu| = \int_E |f| d\mu.$$

- If ν is a signed measure on $(X, \mathcal{M})$ and $P \cup N = X$ is a Hahn decomposition. Then $f = \chi_P - \chi_N$ is extended $|\nu|$-integrable and $\nu(E) = \int_E f d|\nu|$.

4.2 The Radon-Nikodym Theorem

If μ is a measure on $(X, \mathcal{M})$ and $f : X \to [0, \infty)$ is measurable, then $\nu(E) = \int_E f d\mu$ is a measure and we denote $d\nu = f d\mu$. Question: when are two measures μ and ν related like this?

Definition 4.12. Let ν be a signed measure on $(X, \mathcal{M})$ and μ a positive measure on $(X, \mathcal{M})$. ν is **absolutely continuous** w.r.t. μ, denoted $\nu \ll \mu$, if for $E \in \mathcal{M}$: $\mu(E) = 0 \implies \nu(E) = 0$.

Note 4.13. If $d\nu = f d\mu$, then $\nu \ll \mu$.

Proposition 4.14. Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then

$$\nu \ll \mu \iff \text{ for all } \varepsilon > 0 \text{ there is } \delta > 0 \text{ s.t. if } E \in \mathcal{M}, \mu(E) < \delta \text{ then } |\nu(E)| < \varepsilon.$$

Proof. We first reduce to the case of positive ν. Recall $|\nu| = \nu_+ \nu_-$ and let $X = P \cup N$ be a Hahn decomposition for ν. Then $\nu_+(E) = \nu(E \cap P)$ and $\nu_-(E) = -\nu(E \cap N)$. We show that

- We show that $\nu \ll \mu \iff |\nu| \ll \mu$. Indeed

$$\begin{aligned} \nu \ll \mu &\iff (\mu(E) = 0 \implies \nu(E) = 0) \\ &\iff (\mu(E) = 0 \implies \nu(E \cap P) = 0 \text{ and } \nu(E \cap N) = 0) \\ &\iff (\mu(E) = 0 \implies \nu_+(E) = 0 \text{ and } \nu_-(E) = 0) \\ &\iff (\mu(E) = 0 \implies |\nu|(E) = 0) \\ &\iff |\nu| \ll \mu \end{aligned}$$

- We now show that the RHS of the statement of the proposition, S_ν, holds iff $S_{|\nu|}$ holds. On the one hand: if $\mu(E) < \delta$, then $\mu(E \cap P) < \delta$ and $\mu(E \cap N) < \delta$. So

$$\begin{aligned} S_\nu &\implies (\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } E \in \mathcal{M}, \mu(E) < \delta \implies \nu_+(E) < \varepsilon/2 \text{ and } \nu_-(E) < \varepsilon/2) \\ &\implies \underbrace{(\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } E \in \mathcal{M}, \mu(E) < \delta \implies |\nu|(E) < \varepsilon)}_{S_{|\nu|}} \end{aligned}$$

  On the other hand: $|\nu(E)| = |\nu_+(E) - \nu_-(E)| \leq \nu_+(E) + \nu_-(E) = |\nu|(E)$. So

$$S_{|\nu|} \implies (\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } E \in \mathcal{M}, \mu(E) < \delta \implies |\nu(E)| \leq |\nu|(E) < \varepsilon) \iff S_\nu.$$

We now prove the proposition for positive ν.

- $\Longleftarrow$) Let $\varepsilon > 0$. If $\mu(E) = 0$, then $|\nu(E)| < \varepsilon$. And since this holds for all $\varepsilon > 0$, $|\nu(E)| = \nu(E) = 0$. Namely $\nu \ll \mu$.

- $\implies$) Suppose S_ν is false. That is, there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there is $E_n \in \mathcal{M}$ with $\mu(E_n) \leq 1/n$ and $\nu(E_n) \geq \varepsilon$. Then let $F = \bigcap_{n=1}^\infty E_n$ so that by continuity from above $\mu(F) = \lim_{n\to\infty} \mu(E_n) = 0$. And since ν is finite, continuity from above implies $\nu(F) = \lim_{n\to\infty} \nu(E_n) \geq \varepsilon$. So ν is not absolutely continuous w.r.t. μ.

$\square$

Corollary 4.15. Let $f \in L^1(X, \mathcal{M})$. Then for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\mu(E) < \delta \implies \left| \int_E f d\mu \right| < \varepsilon.$$

Proof. Apply Proposition 4.14 to $d\nu = f d\mu$. $\square$

Theorem 4.16. *(Lebesgue-Radon-Nikodym)* Let ν be a σ-finite signed measure, and μ a positive σ-finite measure on $(X, \mathcal{M})$. Then there exist unique σ-finite measures λ and ρ on $(X, \mathcal{M})$ such that

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \text{and} \quad \nu = \lambda + \rho \qquad \text{(Lebesgue)}.$$

Moreover there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$. And any two such functions are equal μ-a.e. (Radon-Nikodym). As notation we write $f = \frac{d\rho}{d\mu}$.

But first a lemma and then a sketch of the proof.

Lemma 4.17. Let μ,ν be positive finite measures. Then either $\mu \perp \nu$ or there is $\varepsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\nu \geq \varepsilon\mu$ on E. Here $\nu \geq \varepsilon\mu$ on E means that E is a positive set for $\nu - \varepsilon\mu$.

Proof. For $n \in \mathbb{N}$, let $X = P_n \cup N_n$ be a Hahn decomposition for $\mu - \frac{1}{n}\nu$. Let $P = \bigcup_{n=1}^{\infty} P_n$ and $N = \bigcap_{n=1}^{\infty} N_n$ so that $N^c = P$. Then N is a negative set for all $\nu - \frac{1}{n}\mu$. In particular $0 \leq \nu(N) \leq \frac{1}{n}\mu(N)$ for all $n \in \mathbb{N}$. So $\nu(N) = 0$. Now if $\mu(P) = 0$, then $\mu \perp \nu$. Otherwise $\mu(P) > 0$ and by continuity there exists $n_0 \in \mathbb{N}$ such that $\mu(P_n) > 0$ for all $n \geq n_0$. Pick $\varepsilon = \frac{1}{n_0}$ and $E = P_{n_0}$. Then $\nu(P_{n_0}) - \frac{1}{n_0}\mu(P_{n_0}) \geq 0$ since $N_{n_0} \cup P_{n_0}$ is a Hahn decomposition. $\square$

As a proof sketch of L-R-N: our goal is to construct $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$ and then define $\lambda = \nu - \rho$ and check $\lambda \perp \mu$. In the case that μ and ν are positive: decompose $X = L \cup M$, with $L \cap M = \emptyset$, $\lambda(M) = 0$ and $\mu(L) = 0$. Then for any $E \in \mathcal{M}$, $\lambda(E) = \lambda(E \cap L) \geq 0$, so

$$\nu(E) = \lambda(E) + \int_E f d\mu \geq \int_E f d\mu.$$

We then define the family

$$\mathcal{F} = \left\{ \varphi : X \to [0, \infty], \text{ measurable and, } \int_E \varphi d\mu \leq \nu(E) \text{ for all } E \in \mathcal{M} \right\}$$

and pick $f \in \mathcal{F}$ by maximizing the mass we put in ρ.

Proof. (Lebesque-Radon-Nikodym) Some quick checks:

- $\mathcal{F} \neq \emptyset$ since $0 \in \mathcal{F}$
- If $\varphi, \psi \in \mathcal{F}$, then $\zeta = \max\{\varphi, \psi\} \in \mathcal{F}$. Indeed let $A = \{x \in X : \varphi(x) > \psi(x)\}$. Then

$$\int_E \zeta d\mu = \int_{E \cap A} \varphi d\mu + \int_{E \setminus A} \psi d\mu \leq \nu(A \cap E) + \nu(E \setminus A) = \nu(E).$$

First suppose that μ, ν are positive and finite and let

$$a = \sup \left\{ \int_X \varphi d\mu : \varphi \in \mathcal{F} \right\} \leq \nu(X) < \infty.$$

There exist $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}$ such that $\lim_{n\to\infty} \int_X \varphi_n d\mu = a$. Let $g_n = \max\{\varphi_1, \ldots, \varphi_n\}$, and let $f = \sup_n g_n$. Then $g_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, $\{g_n\}$ increases to f as $n \to \infty$, and

$$a \geq \int_X g_n d\mu \geq \int_X \varphi_n d\mu \xrightarrow{n\to\infty} a \implies \lim_{n\to\infty} \int_X g_n d\mu = a.$$

So by the M.C.T.

$$\int_E f d\mu = \lim_{n\to\infty}\int_E g_n d\mu \le \nu(E) \qquad \text{and} \qquad \int_X f d\mu = a < \infty \implies 0 \le f < \infty \ \mu\text{-a.e..}$$

In particular $f \in \mathcal{F}$. Now set $d\rho = f d\mu$ and $\lambda = \nu - \rho$. Immediately we have $\nu = \lambda + \rho$, $\rho \ll \mu$, and ρ is positive. λ is also positive since $f \in \mathcal{F}$ implies for all $E \in \mathcal{M}$:

$$\rho(E) = \int_E f d\mu \le \nu(E) \implies \nu(E) - \rho(E) \ge 0.$$

We now check that $\lambda \perp \mu$. Suppose not, then by Lemma 4.16 there is $\varepsilon > 0$ and $E_0 \in \mathcal{M}$ such that $\mu(E_0) > 0$ and $\lambda \ge \varepsilon\mu$ on E_0. Let $d\rho' = \varepsilon\chi_{E_0} d\mu$. Then

$$\rho'(A) = \varepsilon\int_{E_0\cap A} d\mu = \varepsilon\mu(E_0\cap A) \le \lambda(E_0 \cap A).$$

Namely $\rho' \le \lambda = \nu - \rho$ from which it follows that $\rho + \rho' \le \nu$. In other words

$$(f + \varepsilon\chi_{E_0})d\mu \le d\nu \implies f + \varepsilon\chi_{E_0} \in \mathcal{F}.$$

But then $\int_X f + \varepsilon\chi_{E_0} d\mu = a + \varepsilon\mu(E_0) > a$ which contradicts a being the supremum. It remains to check uniqueness. Suppose we have two such decompositions:

$$\nu = \lambda + \rho \qquad \text{and} \qquad \nu = \lambda' + \rho'$$

where $\lambda \perp \mu$, $\rho \ll \mu$ and $\lambda' \perp \mu$, $\rho' \ll \mu$. Furthermore, let $d\rho = f d\mu$ and $\rho = f' d\mu$. Since $\lambda + \rho = \lambda' + \rho'$ we obtain $\lambda - \lambda' = \rho' - \rho$. First of all notice that $(\rho' - \rho) \ll \mu$ since for $E \in \mathcal{M}$ with $\mu(E) = 0$ we have that $\rho(E) = 0$ and $\rho'(E) = 0$. Moreover, $(\lambda - \lambda') \perp \mu$. Indeed: let $X = M \cup L$ with $M \cap L = \emptyset$ and $\mu(L) = 0$, $\lambda(M) = 0$. Define M' and L' similarly for λ'. Then $\mu(L\cup L') \le \mu(L)+\mu(L') = 0$, so $\mu(L\cup L') = 0$. Moreover, $(\lambda-\lambda')((L\cup L')^c) = (\lambda-\lambda')(M\cap M') = 0$ since $M \cap M' \subset M$ and $M \cap M' \subset M'$. Finally, for any $E \in \mathcal{M}$ we can write

$$(\lambda - \lambda')(E) = (\lambda - \lambda')(E \cap (L \cup L')) = (\rho' - \rho)(E \cap (L \cup L')) = 0,$$

since $(\rho' - \rho) \ll \mu$ and $\mu(E \cap (L \cup L')) = 0$. Since this is true for all $E \in \mathcal{M}$ we must have that $\lambda = \lambda'$ and $\rho = \rho'$. Finally, for all $n \in \mathbb{N}$, let $P_n = \{x \in X : f'(x) \ge f(x) + \frac{1}{n}\}$ and $N_n = \{x \in X : f(x) \ge f'(x) + \frac{1}{n}\}$. Then

$$(\rho' - \rho)(P_n) = 0 \implies 0 = \int_{P_n}(f' - f)d\mu \ge \frac{1}{n}\mu(P_n) \implies \mu(P_n) = 0.$$

Similarly for $\mu(N_n) = 0$. So $E_n = P_n \cup N_n = \{x \in X : |f(x) - f'(x)| \ge \frac{1}{n}\}$ are all null sets with $E_n \subset E_{n+1}$. Therefore by continuity from above

$$\mu\left(\{x \in X : f'(x) \ne f(x)\}\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty}\mu(E_n) = 0.$$

That is, $f = f'$ μ-a.e.

Now suppose that μ, ν are positive and σ-finite. We write the disjoint unions $X = \bigcup_{m=1}^{\infty} E_m$ with $\mu(E_m) < \infty$, and $X = \bigcup_{m=1}^{\infty} F_m$ with $\nu(F_m) < \infty$. Then

$$X = \bigcup_{n,m=1}^{\infty}(E_n \cap F_m)$$

is a disjoint union with $\mu(E_n \cap F_m) < \infty$ and $\nu(E_n \cap F_m) < \infty$. For each $n, m \in \mathbb{N}$ let

$$\mu_{n,m}(A) = \mu(A \cap E_n \cap F_m) \qquad \text{and} \qquad \nu_{m,n}(A) = \nu(A \cap E_n \cap F_m).$$

Then $\mu(A) = \sum_{n,m} \mu_{n,m}(A)$ and similarly for ν. By the previous case:

$$d\nu_{n,m} = d\lambda_{n,m} + f_{n,m} d\mu_{n,m}$$

with $\lambda_{n,m} \perp \mu_{n,m}$. It remains to pick

$$\lambda = \sum_{n,m} \lambda_{n,m} \qquad \text{and} \qquad f = \sum_{n,m} f_{n,m} \chi_{E_n \cap F_m}$$

and to verify that $\lambda \perp \mu$.

The general case of signed σ-finite measures follows by applying the previous case to ν_+ and ν_-. $\square$

Example 4.18. Let $F : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable. Then $dm_F = F' dm$, namely $\frac{dm_f}{dm} = \frac{dF}{dx}$ where F' is the derivative of F in the classical sense. Indeed: $m_F((a, b]) = F(b) - F(a)$ and

$$\int_{(a,b]} F' dm = \int_a^b F'(x) dx = F(b) - F(a)$$

by F.T.C. Hence $dm_f = F' dm$ on intervals, and therefore on all of $\mathcal{L}$ by uniqueness.

4.3 Differentiation on $\mathbb{R}^n$

In this section we consider $(X, \mathcal{M}, \mu) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), m^n)$ unless otherwise specified. Consider the motivating example: if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $d\nu = f dm$. Then

$$\begin{aligned} \frac{d\nu}{dm}(x) = f(x) = \frac{d}{dx} \int_a^x f(t) dt = \lim_{r \to 0} \frac{1}{2r} \int_{x-r}^{x+r} f(t) dt \\ = \lim_{r \to \infty} \frac{1}{m((x-r, x+r])} \int_{(x-r,x+r]} f dm = \lim_{r \to \infty} \frac{\nu((x-r, x+r])}{m((x-r, x+r])} \end{aligned}$$

We would like to generalize this to $n \geq 1$, and to ν which are not absolutely continuous with respect to the Lebesgue measure.

Definition 4.19. A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is **locally integrable**, denoted $f \in L^1_{\text{loc}}$, if $\int_K |f| dm < \infty$ for all bounded sets $K \in \mathcal{M}$. For $f \in L^1_{\text{loc}}$ we define its **average** by

$$(A_r f)(x) = \frac{1}{m(B_r(x))} \int_{B_r(x)} f dm = \fint_{B_r(x)} f dm,$$

here $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ is the open ball.

Lemma 4.20. If $f \in L^1_{\text{loc}}$, the map $(0, \infty) \times \mathbb{R}^n \to \mathbb{C}$ given by $(r, x) \mapsto (A_r f)(x)$ is jointly continuous.

Proof. Since $r \mapsto m(B_r(x)) = \omega_n r^n$ is continuous, it suffices to consider

$$(r,x) \mapsto \int_{B_r(x)} f dm = \int f \chi_{B_r(x)} dm.$$

For $y \notin \partial B_r(x)$ we have $\chi_{B_{r_k}(x_k)}(y) \to \chi_{B_r(x)}$ if $(r_k, x_k) \to (r, x)$. So $\chi_{B_{r_k}(x_k)} f \to \chi_{B_r(x)} f$ a.e. Moreover

$$\left|\chi_{B_{r_k}(x_k)} f\right| \le \chi_{B_{r+1}(x)} |f|$$

for large enough k. And $\chi_{B_{r+1}(x)}|f|$ is L^1 because $f \in L^1_{\text{loc}}$. So by the D.C.T.

$$\int \chi_{B_{r_k}(x_k)} f dm \to \int \chi_{B_r(x)} f dm.$$

□

Definition 4.21. Let $f \in L^1_{\text{loc}}$. The **Hardy-Littlewood Maximal function** of f is

$$(Hf)(x) = \sup\{(A_r|f|)(x), r > 0\}.$$

Note 4.22. Hf is measurable since

$$(Hf)^{-1}((a,\infty)) = \bigcup_{\substack{r\in\mathbb{Q} \\ r>0}} (A_r|f|)^{-1}((a,\infty))$$

is open by continuity of $A_r|f|$ (Lemma 4.20).

Theorem 4.23. There is $C > 0$ depending only on n (the spatial dimension) such that for all $\alpha > 0$ and all $f \in L^1$,

$$m\left(\{x \in \mathbb{R}^n : (Hf)(x) > \alpha\}\right) \le \frac{C}{\alpha} \int |f| dm.$$

Note 4.24. This is a strengthening of Markov's Inequality. And the bound is tight in the sense that for $f \in L^1$, $f \neq 0$, we have $m\left(\{Hf > \alpha\}\right) \ge C/\alpha$ for α small enough.

Lemma 4.25. *(Covering lemma)* Let $\mathcal{C}$ be a collection of balls in $\mathbb{R}^n$. Let $U = \bigcup_{B\in\mathcal{C}} B$. For any $0 < c < m(U)$ there is $k \in \mathbb{N}$ and $B_1, \dots, B_k \in \mathcal{C}$ disjoint such that $\sum_{j=1}^k m(B_j) > 3^{-n} c$.

Proof. By inner regularity:

$$m(U) = \sup\{m(K) : K \subset U \text{ is compact}\}.$$

So there is a compact $K \subset U$ such that $c < m(K) < m(U)$. By compactness there is $A_1, \dots, A_\ell \in \mathcal{C}$ such that $\bigcup_{j=1}^{\ell} A_j \supset K$. Let B_1 be the A_j with largest radius. Now recursively take B_{i+1} to be the remaining A_j of largest radius and so that A_j is disjoint from $B_1, \dots, B_i$. Now if $A_{j_0} \notin \{B_1, \dots, B_k\}$, then there is B_j such that $A_{j_0} \cap B_j \neq \emptyset$. Let $B_{\underline{j}}$ be the one of smallest index (largest radius). Then the radius of A_{j_0} is at most the radius of B_j. Hence $A_{j_0} \subset 3B_{\underline{j}}$. So $K \subset \bigcup_{j=1}^k 3B_j$ and hence

$$c < m(K) \le \sum_{j=1}^k m(3B_j) = 3^n \sum_{j=1}^k m(B_j)$$

as desired. □

Proof. (Theorem 4.23) Let $\alpha > 0$ and let $E_\alpha = \{x \in \mathbb{R}^n : (Hf)(x) > \alpha\}$. For $x \in E_\alpha$ there is r_x such that $(A_{r_x} f)(x) > \alpha$. Let $U = \bigcup_{x\in E_\alpha} B_{r_x}(x)$ so that $E_\alpha \subset U$ and let $c < m(E_\alpha) \leq m(U)$. By the covering lemma there is $k \in \mathbb{N}$ and disjoint $\{B_{r_j}(x_j)\}_{j=1}^k$ such that $3^{-n}c < \sum_{j=1}^k m(B_{r_j}(x_j))$. The condition $(A_{r_j}|f|)(x_j) > \alpha$ becomes

$$m(B_{r_j}(x_j)) < \frac{1}{\alpha}\int_{B_{r_j}(x_j)} |f|dm.$$

And hence (since $B_j(x_j)$ are disjoint) we have

$$c < 3^n \frac{1}{\alpha}\sum_{j=1}^k \int_{B_{r_j}(x_j)} |f|dm \leq \frac{3^n}{\alpha}\int |f|dm.$$

The claim follows by taking $c \to m(E_\alpha)$. $\square$

Lemma 4.26. If $f : \mathbb{R}^n \to \mathbb{C}$ is continuous, then $f(x) = \lim_{r\to 0^+}(A_r f)(x)$ for all $x \in \mathbb{R}^n$.

Proof. First of all, $f \in L^1_{\text{loc}}$ since on compact sets continuous f is bounded. Now let $x \in \mathbb{R}^n$ and $\varepsilon > 0$. By continuity there is $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$. For $0 < r < \delta$:

$$|A_r f(x) - f(x)| = \left| \fint_{B_r(x)} (f(y) - f(x))dy \right| \leq \fint_{B_r(x)} |f(y) - f(x)|dy < \varepsilon.$$

$\square$

Proposition 4.27. If $f \in L^1_{\text{loc}}$, then $\lim_{r\to 0^+}(A_r f)(x) = f(x)$ for m-a.e. $x \in \mathbb{R}^n$.

Proof. Let $N \in \mathbb{N}$ and consider the claim on $B_N(0)$. For $|x| < N$ and $r < 1$ we have

$$A_r f(x) = A_r \tilde{f}(x) \text{ where } \tilde{f} = f\chi_{B_{N+1}(0)}$$

so we may consider $f \in L^1$. Let $\varepsilon > 0$. By HW#6 Problem 1 there is a continuous function $g \in L^1$ such that $\int |f - g|dm < \varepsilon$. Now

$$|A_r f(x) - f(x)| \leq |\underbrace{A_r f(x) - A_r g(x)}_{A_r(f-g)(x)}| + |A_r g(x) - g(x)| + |g(x) - f(x)|.$$

Taking the lim sup of both sides (and appealing to Lemma 4.26) yields:

$$\limsup_{r\to 0^+} |A_r f(x) - f(x)| \leq H(f - g)(x) + |f(x) - g(x)|.$$

For $j \in \mathbb{N}$, let $E_j = \left\{x \in B_N(0) : \limsup_{r\to 0^+} |A_r f(x) - f(x)| > \frac{1}{j}\right\}$ and note that by the above inequality:

$$E_j \subset \left\{x : H(f - g)(x) > \frac{1}{2j}\right\} \cup \left\{x : |f(x) - g(x)| > \frac{1}{2j}\right\}.$$

By Markov's Inequality [see HW#8 Problem 2 (iii)] we have $m\left(\left\{x : |f(x) - g(x)| > \frac{1}{2j}\right\}\right) < 2j\varepsilon$ and by the Maximal Theorem $m\left(\left\{x : H(f - g)(x) > \frac{1}{2j}\right\}\right) < 2jC\varepsilon$. Therefore $m(E_j) \leq 2j\varepsilon(1+C)$ and since $\varepsilon > 0$ was arbitrary we conclude $m(E_j) = 0$ for all $j \in \mathbb{N}$. Finally $\lim_{r\to 0^+} A_r f(x) = f(x)$ for all $x \notin \bigcup_{j=1}^\infty E_j$ concluding the proof. $\square$

Definition 4.28. Let $f \in L^1_{\text{loc}}$. Its **Lebesgue set** L_f is

$$L_f = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \fint_{B_r(x)} |f(y) - f(x)| dy = 0 \right\}.$$

Theorem 4.29. If $f \in L^1_{\text{loc}}$, then $m(L_f^c) = 0$.

Proof. Apply the previous theorem to $|f(x) - \lambda|$ for any $\lambda \in \mathbb{C}$:

$$(\star) \qquad \lim_{r \to 0^+} \fint_{B_r(x)} |f(y) - \lambda| dy = |f(x) - \lambda| \quad \text{ for all } x \in E_\lambda^c \text{ with } m(E_\lambda) = 0.$$

Let Λ be a countable dense set in $\mathbb{C}$ and $E = \bigcup_{\lambda \in \Lambda} E_\lambda$ so that $m(E) = 0$. If $x \in E^c = \bigcap_{\lambda \in \Lambda} E_\lambda^c$ we pick $\lambda \in \Lambda$ such that $|f(x) - \lambda| < \varepsilon$. Then

$$|f(y) - f(x)| \leq |f(y) - \lambda| + |f(x) - \lambda| < |f(y) - \lambda| + \varepsilon.$$

Hence

$$\limsup_{r \to 0^+} \fint_{B_r(x)} |f(y) - f(x)| dy \overset{(\star)}{\leq} |f(x) - \lambda| + \varepsilon < 2\varepsilon$$

which concludes the proof since $\varepsilon > 0$ was arbitrary. $\square$

Definition 4.30. A family of Borel sets $\{E_r\}_{r>0}$ **shrinks nicely to** x if $E_r \subset B_r(x)$ and there is $\alpha > 0$ such that $M(E_r) \geq \alpha m(B_r(x))$ for all $r > 0$. Note that x need not be in E_r.

Corollary 4.31. *(Lebesgue differentiation Theorem)* Let $f \in L^1_{\text{loc}}$ and $x \in L_f$. If $\{E_r\}_{r>0}$ shrinks nicely to x then

$$\lim_{r \to 0^+} \fint_{E_r} |f(y) - f(x)| dy = 0.$$

In particular there is convergence for m-a.e. x.

Proof. $\displaystyle \frac{1}{m(E_r)} \int_{E_r} |f(x) - f(y)| dy \leq \frac{1}{\alpha m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dy \xrightarrow{r \to 0^+} 0.$ $\square$

Example 4.32. Let $f \in L^1_{\text{loc}}$ and $F(x) = \int_{[a,x]} f dm$. Then

$$\lim_{h \to 0^+} h^{-1}(F(x+h) - F(x)) - f(x) = \lim_{h \to 0^+} \frac{1}{m(E_h)} \int_{E_h} (f(y) - f(x)) dy = 0 \qquad \text{a.e.}$$

since $E_h = (x, x+h)$ shrinks nicely to x. Can do the same thing for $\lim_{h \to 0^-}$ with $E_h = (x+h, x)$.

Proposition 4.33. *(FTC)* Let $f \in L^1_{\text{loc}}$ and $F(x) = \int_{[a,x]} f dm$. Then F is differentiable m-a.e. with $F'(x) = f(x)$ for a.e. x

Example 4.34. *(Motivating example)* For $\nu = \delta_{x_0}$ on $\mathbb{R}$ we have

$$\lim_{r \to 0^+} \frac{\nu(B_r(x))}{m(B_r(x))} = \begin{cases} \infty & \text{if } x = x_0 \\ 0 & \text{o.w.} \end{cases}.$$

In particular the limit equals zero m-a.e.

Definition 4.35. A Borel measure ν on $\mathbb{R}^n$ is said to be **regular** if

(i) $\nu(K) < \infty$ for compact $K \subset \mathbb{R}^n$

(ii) $\nu(E) = \inf\{\nu(U) : U \supset E, \text{ and } U \text{ open}\}$ for all measurable E *(outer regularity)*

Note 4.36.

- In fact, (i) $\implies$ (ii)
- Regular measures are σ-finite since $\mathbb{R}^n$ can be covered by compact sets
- If ν is signed or complex, then ν is regular if $|\nu|$ is regular

Example 4.37.

- Any Lebesgue-Stieltjes measure is regular
- If $f \in L^+$ and $d\nu = f dm$, then $f \in L^1_{\text{loc}}$ if and only if ν is regular

Theorem 4.38. Let ν be a regular signed or complex measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and $d\nu = d\lambda + f dm$ be its Lesbesgue decomposition. Then

$$\lim_{r \to 0^+} \frac{\nu(E_r)}{m(E_r)} = f(x) \qquad \text{for } m\text{-a.e. } x, \text{ and where } \{E_r\}_{r>0} \text{ shrinks nicely to } x.$$

Proof. Since $d\lambda + f dm$ is regular and since $d\lambda$ and $f dm$ are mutually singular, we have that $d\lambda$ and $f dm$ are regular. In particular, $f \in L^1_{\text{loc}}$ and so by the Lebesgue differentiation Theorem it suffices to check that $\lambda(E_r)/m(E_r) \to 0$ as $r \to 0^+$ for m-a.e. x. Moreover,

$$\left|\frac{\lambda(E_r)}{m(E_r)}\right| \le \frac{|\lambda|(E_r)}{m(E_r)} \le \frac{|\lambda|(B_r(x)}{\alpha m(B_r(x))}$$

so it suffices to consider λ positive and $B_r(x)$ in place of E_r. Since $\lambda \perp m$, there is $A \in \mathcal{B}(\mathbb{R}^n)$ such that $\lambda(A) = m(A^c) = 0$ so it suffices to consider $x \in A$ (since we only seek m-a.e. convergence). For $k \in \mathbb{N}$ let

$$F_k = \left\{x \in A : \limsup_{r \to 0^+} \frac{\lambda(B_r(x))}{m(B_r(x))} > \frac{1}{k}\right\}$$

we claim that $m(F_k) = 0$, which would conclude the proof since then $m\left(\bigcup_{k\in\mathbb{N}} F_k\right) = 0$ and on the complement: $\left(\bigcup_{k\in\mathbb{N}} F_k\right)^c = \bigcap_{k\in\mathbb{N}} F_k^c$ we have that $\limsup_{r\to 0^+} \frac{\lambda(B_r(x))}{m(B_r(x))} = 0$. Indeed, let $\varepsilon > 0$. Since $\lambda(A) = 0$ regularity implies there is an open $U \supset A$ such that $\lambda(U) < \varepsilon$, notice that $F_k \subset A \subset U$. By definition of F_k, for any $x \in F_k$ there is $r_x > 0$ such that $B_{r_x}(x) \subset U$ and more importantly that $\lambda(B_{r_x}(x)) > \frac{1}{k} m(B_{r_x}(x))$. Let $V = \bigcup_{x\in F_k} B_{r_x}(x)$ so that $F_k \subset V$. By the covering lemma for any $c < m(V)$ there are disjoint $B_{r_1}(x_1), \ldots, B_{r_J}(x_J)$ with

$$c < 3^n \sum_{j=1}^{J} m(B_{r_j}(x_j)) < 3^n k \sum_{j=1}^{J} \lambda(B_{r_j}(x_j)) \le 3^n k \lambda(V) \le 3^n k \lambda(U) < 3^n k\varepsilon.$$

Hence $m(F_k) \le m(V) \le 3^n k\varepsilon$. $\square$

4.4 Differentiation on $\mathbb{R}$

Now we let $n = 1$, i.e. $x \in \mathbb{R}$, and consider Lebesgue-Stieltjes measures. A question: for which F does $F(x) - F(a) = \int_a^x F'(t)dt$? Necessary conditions are F' exists a.e., and $F' \in L^1_{\text{loc}}$, but are these also sufficient? Note that if $F : \mathbb{R} \to \mathbb{R}$ is increasing, then

$$F(x^-) = \sup_{y<x} F(y) \leq \inf_{y>x} F(y) = F(x^+).$$

Proposition 4.39. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing, then

(i) F has at most countably many discontinuities

(ii) Let $G(x) = F(x^+)$. Then $G = F$ a.e., F and G are both differentiable a.e., and $G' = F'$ a.e.

Proof.

(i) Since $x < y \implies F(x^+) \leq F(y^-)$, the intervals $I_x = (F(x^-), F(x^+))$ are disjoint (they could be $\emptyset$ if F is continuous at x). Let $P = \{x \in \mathbb{R} : I_x \neq \emptyset\}$. For any $x \in P$, pick $q_x \in I_x \cap \mathbb{Q}$. Since I_x are disjoint, the map $P \ni x \mapsto .q_x \in \mathbb{Q}$ is injective, so P is countable.

(ii) G is increasing and right-continuous. So we can consider m_G which is regular (as a Lebesgue-Stieltjes measure). Hence Theorem 4.38 implies $\lim_{r\to 0^+} \frac{m_G(E_r)}{m(E_r)}$ exists m-a.e. for E_r shrinking nicely to x. Take $E_r = (x, x + r]$ and compute $\frac{m_G(E_r)}{m(E_r)} = \frac{G(x+r)-G(x)}{r}$ (do the same for $E_r = (x-r, x])$ to conclude that G' exists a.e. Now let $H = G-F$. By definition $H \geq 0$, and by (i) $\{x \in \mathbb{R} : H(x) \neq 0\}$ is countable, so enumerate it as $\{x_j : j \in \mathbb{N}\}$. Define the measure

$$\mu = \sum_{j=1}^{\infty} H(x_j)\delta_{x_j},$$

which is regular since

$$\mu([-N, N]) = \sum_{|x_j|<N} H(x_j) = \sum_j F(x_j^+) - F(x_j) \leq F(N) - F(-N) < \infty.$$

And $\mu \perp m$ since $m(\{x_j : j \in \mathbb{N}\}) = 0$. Hence by Theorem 4.38

$$|h^{-1}(H(x+h)-H(x)| \leq |h|^{-1}\mu((x-2|h|, x+2|h|)) = 4\frac{\mu((x-2|h|, x+2|h|))}{m((x-2|h|, x+2|h|))} \xrightarrow{h\to 0} 0 \qquad \text{a.e.}$$

Namely, $H' = 0$ a.e., and since $H = G - F$ we conclude F' exists a.e. and is equal to G' a.e.

□

If $f \in L^1_{\text{loc}} \cap L^+$, then $x \mapsto \int_a^x f(t)dt$ defines an increasing function, and hence Proposition 4.39 applies. Extending this to complex value f leads to the following definition.

Definition 4.40. Let $F : \mathbb{R} \to \mathbb{C}$.

(i) The **total variation function** of F is

$$T_F(x) = \sup \left\{ \sum_{j=1}^{N} |F(x_j) - F(x_{j-1})| : -\infty < x_0 < x_1 < \cdots < x_N = x, N \in \mathbb{N} \right\}$$

(ii) Let $a < b$. The **total variation of F on** $[\mathbf{a}, \mathbf{b}]$ is $T_F(b) - T_F(a)$.

(iii) F is of **bounded variation**, denoted $F \in \mathrm{BV}$, if

$$\lim_{x \to \infty} T_F(x) < \infty.$$

(iv) F is of **bounded total on** $[\mathbf{a}, \mathbf{b}]$, denoted $F \in \mathrm{BV}([a, b])$, if $T_F(b) - T_F(a) < \infty$.

Note 4.41.

(i) If $F : \mathbb{R} \to \mathbb{R}$ is increasing, then

$$\sum_{j=1}^{N} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{N} (F(x_j) - F(x_{j-1})) = F(b) - F(a),$$

hence $F \in \mathrm{BV}([a, b])$ and $F \in \mathrm{BV}$ whenever F is bounded.

(ii) Let $F : \mathbb{R} \to \mathbb{C}$. If F is differentiable with bounded derivative, then $F \in \mathrm{BV}([a, b])$ since

$$\sum_{j=1}^{N} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{N} |F(x_j^\star)|(x_j - x_{j-1}) \leq C(b - a),$$

but in general $F \notin \mathrm{BV}$.

Lemma 4.42. Let $F : \mathbb{R} \to \mathbb{R}$ is BV, then $T_F \pm F$ are increasing.

Proof. Let $x < y$, let $\varepsilon > 0$. There is $N \in \mathbb{N}$ and $x_0 < x_1 < \cdots < x_N = x$ such that

$$\sum_{j=1}^{N} |F(x_j) - F(x_{j-1})| \geq T_F(x) - \varepsilon.$$

Adding y to this partition yields a new partition, and since $T_F(y)$ is a supremum over such partitions:

$$T_F(y) \geq |F(y) - F(x)| + \sum_{j=1}^{N} |F(x_j) - F(x_{j-1})| \geq T_F(x) + |F(y) - F(x)| - \varepsilon.$$

Equivalently, and since $\varepsilon > 0$ was arbitrary,

$$T_F(x) - T_F(y) \overset{(1)}{\leq} F(y) - F(x) \overset{(2)}{\leq} T_F(y) - T_F(x).$$

Hence

$$T_F(y) + F(y) \overset{(1)}{\geq} T_F(x) + F(x) \qquad \text{and} \qquad T_F(y) - F(y) \overset{(2)}{\geq} T_F(x) - F(x).$$

$\square$

Definition 4.43. Let $F : \mathbb{R} \to \mathbb{R}$ be BV. Then the functions $F_{\pm} := \frac{1}{2}(T_F \pm F)$ are the **positive/negative variations** of F . The **Jordan Decomposition** of F is $F = F_+ - F_-$.

Note 4.44. For $F : \mathbb{R} \to \mathbb{C}$, F is BV is $\text{Re}(F), \text{Im}(F) \in \text{BV}$ and

$$F = (\text{Re}F)_+ - (\text{Re}F)_- + i((\text{Im}F)_+ - (\text{Im}F)_-)$$

Proposition 4.45. Let $F \in \text{BV}$. Then

(i) The limits $F(x^{\pm})$, $F(\pm\infty)$ exist

(ii) F has at most countably many discontinuities

(iii) F is differentiable a.e.

(iv) $G(x) = F(x^+)$ is differentiable a.e. and $G' = F'$ a.e.

Proof. Apply Proposition 4.39 to the Jordan decomposition. □

Definition 4.46.

$$\text{NBV} = \{F \in \text{BV} : F \text{ is right-continuous and } F(-\infty) = 0\}.$$

Note 4.47.

- A complex measure is always finite
- If $F \in \text{BV}$ and $F(-\infty) > -\infty$, then $G(x) = F(x^+) - F(-\infty)$ is NBV

Theorem 4.48.

(i) If ν is a complex Borel measure, then $F(x) = \nu((-\infty, x])$ is NBV

(ii) If $F \in \text{NBV}$, there is a unique Borel measure m_F such that $m_F((-\infty, x]) = F(x)$.

Proof. Skipped, but see Proposition 1.24 and use the Jordan Decomposition. □

Putting everything together, let $F \in \text{NBV}$ and let $dm_F = d\lambda + f dm$ be its L-R-N decomposition. By the differentiation theorem:

$$F'(x) = \lim_{r \to 0^+} \frac{m_F(E_r)}{m(E_r)} \qquad \text{for a.e. } x,$$

where $E_r = (x, x+r]$ is a family of sets that shrinks nicely to x. In fact, one can prove the following:

Theorem 4.49. Let $F \in \text{NBV}$. Then

(i) F' exists a.e. and $F' \in L^1$

(ii) $m_F \perp m$ if and only if $F' = 0$ a.e.

(iii) $m_F \ll m$ if and only if $F(x) = \int_{-\infty}^{x} F'(t)dt$

Definition 4.50. A function $F : \mathbb{R} \to \mathbb{C}$ is **absolutely continuous** (AC) if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $(a_1, b_1), \ldots, (a_N, b_N)$ are disjoint and $\sum_{j=1}^N (b_j - a_j) < \delta$, then $\sum_{j=1}^N |F(b_j) - F(a_j)| < \varepsilon$.

Note 4.51. If $F \in \mathrm{AC}$, then F is uniformly continuous (take $N = 1$ in the above).

Proposition 4.52. Let $F \in \mathrm{NBV}$. Then $F \in \mathrm{AC}$ if and only if $m_F \ll m$.

Proof. Assume $m_F \ll m$, then $F \in \mathrm{AC}$ by Proposition 4.14, with $E = \bigcup_{i=1}^N (a_i, b_i]$. Now assume that $F \in \mathrm{AC}$ and let E be a measurable set such that $m(E) = 0$. Let $\varepsilon > 0$ and δ as in the definition of absolute continuity. By the regularity of m, there is an open set $U_1 \supset E$ such that $m(U_1) < \delta$. And by regularity of m_F, there are open sets $U_1 \supset U_2 \supset \cdots \supset E$ such that $\lim_{j\to\infty} m_F(U_j) = m_F(E)$. Since an open set is equal to a countable disjoint union of open intervals we can write $U_j = \bigcup_{k=1}^\infty (a_j^k, b_j^k)$. And for any $N \in \mathbb{N}$:

$$\sum_{j=1}^N (b_j^k - a_j^k) \leq m(U_j) \leq m(U_1) < \delta.$$

So by the absolute continuity of F, and since F is continuous:

$$\varepsilon > \sum_{j=1}^N |F(b_j^k) - F(a_j^k)| = \sum_{j=1}^N |m_F((a_j^k, b_j^k])| = \sum_{j=1}^N |m_F((a_j^k, b_j^k))|.$$

Letting $N \to \infty$, and then $j \to \infty$ (with continuity from above) yields:

$$|m_F(E)| = \lim_{j\to\infty} |m_F(U_j)| = \lim_{j\to\infty} \left| \sum_{k=1}^\infty m_F((a_j^k, b_j^k)) \right| = \lim_{j\to\infty} \lim_{N\to\infty} \sum_{j=1}^N |m_F((a_j^k, b_j^k))| < \varepsilon.$$

Then since $\varepsilon > 0$ was arbitrary, we conclude that $m_F(E) = 0$ and so $m_F \ll m$. $\square$

To summarize: for $F \in \mathrm{NBV}$, then

$$F \in \mathrm{AC} \iff m_F \ll m \iff F(x) = \int_{-\infty}^x F'(t)dt.$$

On bounded intervals we can do even better.

Theorem 4.53. Let $F : [a, b] \to \mathbb{C}$. Then the following are equivalent.

(i) $F \in \mathrm{AC}([a, b])$

(ii) $F(x) - F(a) = \int_a^x f(t)dt$ for some $f \in L^1([a, b])$

(iii) F is differentiable a.e., $F' \in L^1([a, b])$ and $F(x) - F(a) = \int_a^x F'(t)dt$

Proof.

- (i) $\implies$ (iii): We show that if $F \in \mathrm{AC}([a,b])$, then $F \in \mathrm{BV}([a,b])$. Let $\varepsilon = 1$, and $\delta > 0$ be as in the definition of absolute continuity. Let $N = \lfloor \delta^{-1}(b-a)+1 \rfloor$ and let $a = x_0 < x_1 < \cdots < x_N = b$. By possibly adding points the partition of $[a,b]$, we obtain N groups of disjoint intervals each of length less than δ. So by absolute continuity: $\sum |F(x_j) - F(x_{j-1})| \leq N$, and since the partition is arbitrary we conclude that $F \in \mathrm{BV}([a,b])$. Now define

$$\tilde{F}(x) = \begin{cases} 0 & x < a \\ F(x) - F(a) & x \in [a,b] \\ F(b) - F(a) & x > b \end{cases}.$$

  Then $\tilde{F} \in \mathrm{NBV}$ and the claim follows from the previous result.

- (iii) $\implies$ (ii): Immediate.

- (ii) $\implies$ (i): We extend f by 0 outside $[a,b]$ and extend F same as before. Then $f \in L^1(\mathbb{R})$ so $d\nu = f dm$ is a complex Borel measure and $\nu = m_{F-F(a)} \ll m$. Then by previous result $F - F(a) \in \mathrm{AC}$ hence (i) holds.

$\square$

Note 4.54. Let C be the Cantor set and let F be the Cantor function.

- Then $F'(x) = 0$ for $x \in [0,1] \setminus C$. That is $F' = 0$ a.e., and so the F.T.C. fails since $\int_0^x F'(t)dt = 0 \neq F(x)$. So F is not absolutely continuous (but note that F is uniformly continuous).

- Also, $F' = 0$ a.e. implies $m_F \perp m$. But notice that $m_F(\{x\}) = 0$ for all $x \in [0,1]$ since F is continuous. This is an example of a **singular continuous measure**

Definition 4.55. A Borel measure μ on $\mathbb{R}$ is

- **discrete** if $\mu = \sum_j c_j |\delta_{x_j}|$ and $\sum_j |c_j| < \infty$

- **continous** if $\mu(\{0\}) = 0$ for all $x \in \mathbb{R}$

Lemma 4.56. Let μ be a complex Borel measure. Then the set $E = \{x \in \mathbb{R} : \mu(\{x\}) \neq 0\}$ is at most countable.

Proof. $\mu(E) = \sum_{x \in E} \mu(\{x\}) < \infty$ since complex measures are finite. Hence E is at most countable. $\square$

Hence $\mu(A) = \mu(A \cap E) + \mu(A \cap E^c) =: \mu_d(A) + \mu_c(A)$ yields a decomposition of any complex Borel measure into a discrete and continuous part. By definition $\mu_d \perp m$. For μ_c we don't know, but we can apply L-R-N to obtain the following decomposition:

$$\mu = \mu_d + \mu_{ac} + \mu_{sc}$$

where $\mu_d \perp m$ is the discrete part of μ, $\mu_{sc} \perp m$ is the **singular continuous** part of μ, and $\mu_{ac} \ll m$ is the absolutely continuous part of μ

Appendices

A L^p spaces

Definition A.1. Let $1 \le p \le \infty$. Let $(X, \mathcal{M}, \mu)$ be a measure space.

- If $p < \infty$,
$$\mathcal{L}^p = \left\{ \psi : X \to \mathbb{C} \text{ such that } \psi \text{ is measurable and } \int |\psi|^p d\mu < \infty \right\}$$
and
$$\|\psi\|_p = \left(\int |\psi|^p d\mu \right)^{1/p}$$

- If $p = \infty$,
$$\mathcal{L}^\infty = \left\{ \psi : X \to \mathbb{C} \text{ such that } \psi \text{ is measurable and } \operatorname*{ess\,sup}_{x \in X} |\psi(x)| < \infty \right\}$$
and
$$\|\psi\|_\infty = \operatorname*{ess\,sup}_{x \in X} |\psi(x)| \qquad \text{where } \operatorname*{ess\,sup}_{x \in X} |\psi(x)| = \inf\{M \ge 0 : |\psi(x)| \le M \mu\text{-a.e.}\}$$
Note that for any $n \in \mathbb{N}$, there is N_n with $\mu(N_n) = 0$ such for all $x \in N_n^c$
$$|\psi(x)| \le \|\psi\|_\infty + \frac{1}{n}.$$
Let $N = \bigcup_{n \in \mathbb{N}} N_n$. Then N is null and $|\psi(x)| \le \|\psi\|_\infty$ for all $x \in N^c$.

Example A.2. For any $1 \le p \le \infty$, $\mathcal{L}^p$ is a vector space. Some examples:

(i) Let $n \in \mathbb{N}$, $X = \{1, 2, \dots, n\}$, $\mathcal{M} = \mathcal{P}(X)$, and μ the counting measure. Then $\psi : X \to \mathbb{C}$ is identified with the vector $\begin{pmatrix} \psi(1) & \cdots & \psi(n) \end{pmatrix}^\top \in \mathbb{C}^n$ and
$$\int_X |\psi|^2 d\mu = \sum_{i=1}^n |\psi(i)|^2$$
is the squared Euclidean norm.

(ii) Let $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$, and μ the counting measure. We denote $\mathcal{L}^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu) = \ell^p$. Here $\psi : X \to \mathbb{C}$ is identified with the sequence $\{\psi_i\}_{i \in \mathbb{N}}$. And
$$\ell^p = \left\{ \{\psi_i\}_{i \in \mathbb{N}} : \sum_{i=1}^\infty |\psi_i|^p < \infty \right\} \qquad \text{and} \qquad \ell^\infty = \left\{ \{\psi_i\}_{i \in \mathbb{N}} : \sup_{i \in \mathbb{N}} |\psi_i| < \infty \right\}.$$

Definition A.3. A **Banach space** is a complete normed vector space.

- A norm on a vector space V is a map $\|\cdot\| : V \to [0, \infty)$ such that
  (i) $\|v\| = 0 \iff v = 0$
  (ii) $\|\alpha v\| = |\alpha| \|v\|$, for $\alpha \in \mathbb{C}$ and $v \in V$
  (iii) $\|v + w\| \le \|v\| + \|w\|$
- A normed vector space is complete if every Cauchy sequence is convergent
- In a normed vector space, $d(v, w) = \|v - w\|$ is a metric

Note A.4. In the above definition, (i) fails for $\mathcal{L}^p$. The solution is to define the equivalence relation

$$\psi \sim \phi \text{ if } \psi = \phi \text{ a.e.}$$

Then $[\psi] = \{\phi \in \mathcal{L}^p : \phi \sim \psi\}$.

Definition A.5. $L^p(X, \mathcal{M}, \mu)$ is the set $\{[\psi] : \psi \in \mathcal{L}^p(X, \mathcal{M}, \mu)\}$ equipped with the operations

- $[\psi] + [\phi] = [\psi + \phi]$
- $\alpha[\psi] = [\alpha\psi]$
- $\|[\psi]\|_p = \|\psi\|_p$

Lemma A.6. The above operations are well-defined and $L^p(X, \mathcal{M}, \mu)$ is a normed vector space.

Proof.

- "+" is well defined: we need to show that

  $$(\psi_1 \sim \psi_2 \ \&\ \phi_1 \sim \phi_2) \implies \psi_1 + \phi_1 \sim \psi_2 + \phi_2.$$

  Let $N_\psi = \{x \in X : \psi_1(x) \leq \psi_2(x)\}$ and $N_\phi = \{x \in X : \phi_1(x) \neq \phi_2(x)\}$. Then $\mu(N_\psi) = \mu(N_\phi) = 0$, and

  $$\{x \in X : \psi_1(x) + \phi_1(x) \neq \psi_2(x) + \phi_2(x)\} \subset N_\psi \cup N_\phi.$$

  Finally, since $\mu(N_\psi \cup N_\phi) = 0$, we conclude $\psi_1 + \phi_1 \sim \psi_2 + \phi_2$ as desired.
- Similar for well-definedness of scalar multiplication
- The norm is well-defined since integrals of a.e. equal functions are equal
- The vector space axioms are immediate with $0 = [0]$ being the class of functions that are equal to 0 a.e.
- The first two norm axioms are quick to prove. The triangle inequality takes a bit of work (see HW 10)

$\square$

Proposition A.7. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $\psi \in L^\infty$. Then $\psi \in L^p$ for all $1 \leq p \leq \infty$ and

$$\|\psi\|_\infty = \lim_{p\to\infty} \|\psi\|_p.$$

This is one of many results that essentially says: "the infinity norm is the limit of the p norm whenever it makes sense".

Proof. Let $X_r = \{x \in X : |\psi(x)| \geq r\}$. If $\mu(X_r) > 0$ then

$$\liminf_{p\to\infty} \|\psi\|_p \geq \liminf_{p\to\infty} \left(\int_{X_r} |\psi|^p d\mu\right)^{1/p} \geq r \liminf_{p\to\infty} \mu(X_r)^{1/p} = r$$

and

$$\limsup_{p\to\infty} \|\psi\|_p \le \|\psi\|_\infty \limsup_{p\to\infty} \mu(X)^{1/p} = \|\psi\|_\infty.$$

Now pick $r = \|\psi\|_\infty - \varepsilon$ so that $\mu(X_r) > 0$ and so that $\liminf_{p\to\infty} \|\psi\|_p \ge \|\psi\|_\infty - \varepsilon$ for all $\varepsilon > 0$. Thus altogether

$$\|\psi\|_\infty \le \liminf_{p\to\infty} \|\psi\|_p \le \limsup_{p\to\infty} \|\psi\|_p \le \|\psi\|_\infty.$$

So all inequalities are equalities and

$$\liminf_{p\to\infty} \|\psi\|_p = \limsup_{p\to\infty} \|\psi\|_p = \lim_{p\to\infty} \|\psi\|_p = \|\psi\|_\infty.$$

□

Theorem A.8. *(Hölder's Inequality)* Let $1 \le p, q \le \infty$.

(i) If $\frac{1}{p} + \frac{1}{q} = 1$ and $\psi \in L^p$, $\phi \in L^q$, then $\psi\phi \in L^1$ and $\|\psi\phi\|_1 \le \|\psi\|_p \|\phi\|_q$

(ii) If $1 \le r < \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and $\psi \in L^p$, $\phi \in L^q$, then $\psi\phi \in L^r$ and $\|\psi\phi\|_r \le \|\psi\|_p \|\phi\|_q$

Proof.

(i) See HW 10

(ii) Follows from (i) applied to $|\psi|^r$, $|\phi|^r$ and $\frac{p}{r}$, $\frac{q}{r}$.

$$\|\psi\phi\|_r^r = \||\psi|^r |\phi|^r\|_1 \overset{(i)}{\le} \||\psi|^r\|_{p/r} \||\phi|^r\|_{q/r} = \|\psi\|_p^r \|\phi\|_q^r.$$

□

Corollary A.9. Let $1 \le p < q \le \infty$. If $\psi \in L^p \cap L^q$, then $\psi \in L^r$ for all $r \in [p, q]$ and

$$\|\psi\|_{p_\theta} \le \|\psi\|_p^{1-\theta} \|\psi\|_q^\theta,$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p} + \frac{\theta}{q}$ for $\theta \in [0, 1]$.

Proof. ($q < \infty$) By Hölder's Inequality:

$$\|\psi\|_{p_\theta}^{p_\theta} = \int_X |\psi|^{(1-\theta)p_\theta} |\psi|^{\theta p_\theta} d\mu = \||\psi|^{(1-\theta)p_\theta} |\psi|^{\theta p_\theta}\|_1 \le \||\psi|^{(1-\theta)p_\theta}\|_{p/(p_\theta(1-\theta))} \||\psi|^{\theta p_\theta}\|_{q/\theta p_\theta}.$$

We should just check that $\frac{(1-\theta)p_\theta}{p} + \frac{\theta p_\theta}{q} = 1$ (true by assumption), and that

$$|\psi|^{(1-\theta)p_\theta} \in L^{p/(p_\theta(1-\theta))}, \qquad |\psi|^{\theta p_\theta} \in L^{q/\theta p_\theta}.$$

And indeed

$$\||\psi|^{(1-\theta)p_\theta}\|_{p/(p_\theta(1-\theta))} = \left(\int_X \left(|\psi|^{(1-\theta)p_\theta}\right)^{\frac{p}{(1-\theta)p_\theta}}\right)^{\frac{(1-\theta)p_\theta}{p}} = \|\psi\|_p^{(1-\theta)p_\theta},$$

and similarly

$$\||\psi|^{\theta p_\theta}\|_{q/\theta p_\theta} = \|\psi\|_q^{\theta p_\theta}$$

from which the claim follows. □

Theorem A.10. *(Reisz-Fischer)* Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then $L^p(X, \mathcal{M}, \mu)$ is a Banach space.

Lemma A.11. A normed vector space is complete if and only if every absolutely convergent series is convergent.

Proof. (Reisz-Fischer) From what we have already shown about $L^p(X, \mathcal{M}, \mu)$, it suffices to check completeness.

- Case $1 \leq p < \infty$: Let $\{\psi_i\}_{i \in \mathbb{N}}$ be absolutely convergent, namely $\sum_i \|\psi_i\|_p = M < \infty$. By Lemma A.11 it suffices to show that $\sum_i \psi_i$ converges in L^p. Let $G_n = \sum_{i=1}^n |\psi_i(x)|$, this increases point-wise to $G = \sum_i |\psi_i|$ (it may be ∞). Now by the triangle inequality:

$$\|G_n\|_p \leq \sum_{i=1}^n \|\psi_i\|_p \leq M < \infty.$$

  Then by the M.C.T., $G \in L^p$ and $\int_X G^p d\mu = \lim_{n\to\infty} \int |G_n|^p d\mu \leq M^p$. In particular $G(x)$ is finite μ-a.e. Hence there is a null set N such that the numerical series $\sum_i \psi_i(x)$ converges absolutely for $x \in N^c$. Now by completeness of $\mathbb{C}$:

$$S_n(x) = \sum_{i=1}^n \psi_i(x)\chi_{N^c}(x) \to S(x)$$

  for all $x \in X$. Altogether $|S_n(x) - S(x)|^p \to 0$ as $N \to \infty$ and $|S_n(x) - S(x)|^p \leq (2G(x))^p \in L^1$, so the D.C.T. implies

$$\lim_{n\to\infty} \int_X |S_n(x) - S(x)|^p d\mu = 0.$$

  Namely $S = \lim_{n\to\infty} S_n$ in L^p and so the series is convergent.

- Case $p = \infty$: Let $\{\psi_i\}_{i \in \mathbb{N}}$ be a Cauchy sequence in L^∞. Then, by definition of $\|\cdot\|_\infty$, for each $j, k \in \mathbb{N}$ there is a null set $N_{j,k}$ such that

$$|\psi_j(x) - \psi_k(x)| \leq \|\psi_j - \psi_k\|_\infty$$

  for all $x \in N_{j,k}^c$. The set $N = \bigcup_{j,k \in \mathbb{N}} N_{j,k}$ is again a null set. Let $x \in N^c$. Then $\{\psi_i(x)\}_{i \in \mathbb{N}}$ is Cauchy and hence convergent (by the completeness of $\mathbb{C}$), say to $\psi(x)$. It follows that $\psi_j \to \psi$ uniformly on N^c. Namely $\psi_j \to \psi$ in L^∞.

□